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RELATIVE EQUILIBRIA OF A RIGID SATELLITE

IN A CENTRAL GRAVITATIONAL FIELD

DISSERTATION

Jeffrey A. Beck
Major, USAF

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DISSERTATION

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To my unborn baby girl ...

Jeffrey A. Beck

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Abstract

We apply noncanonical Hamiltonian methods to examine relative equilibria of a rigid body in a central gravitational field. These equilibria correspond to fixed points of a reduced set of equations expressed in a rotating frame and are representative of an orbiting satellite with fixed attitude relative to an observer rotating at the orbital rate. Our objective is to clarify the relationship between the classical approximation and a recent noncanonical Hamiltonian treatment. In contrast to the classical approximation, the orbital and attitude equations of motion for the noncanonical system remain coupled and the general solution is a circular orbit for which the orbit center and the center of attraction are not necessarily coincident. Our approach involves development of a hierarchy of Hamiltonian approximations. The hierarchy consists of the existing noncanonical system and two noncanonical formulations which we derive for rigid bodies subject to certain constraints — motion about a fixed point and motion about a point following a Keplerian orbit. The classical solution is dynamically equivalent to this latter constrained (Keplerian) system. We apply Hamiltonian methods to identify relative equilibria and determine stability conditions. In general, we find that relative equilibria for the Keplerian and unconstrained systems are in close agreement.

RELATIVE EQUILIBRIA OF A RIGID SATELLITE IN A CENTRAL GRAVITATIONAL FIELD

I. Introduction

1.1 Background

The motion of a system of particles or point masses acting under their mutual gravitational attraction is the archetypal problem of celestial mechanics. For n particles, the problem has $3n$ degrees of freedom associated with the translation of the particles. The equations of motion form a $6n$th-order system of differential equations. This system of equations has ten first integrals: the total system energy, the three components of the angular momentum about the center of mass, the three components of total linear momentum, and the three components of position of the center of mass at epoch. These integrals are sufficient to ensure integrability only in the case of one or two particles. For a larger number of particles, consideration is often restricted to special cases (*e.g.*, planar motion, central configurations). For a discussion of this problem in the classical Hamiltonian setting, see Meyer and Hall [71] or Pollard [80].

An analysis which treats all bodies as particles may be sufficient if we are only interested in the orbits of bodies which remain separated by distances which are much greater than the dimensions of the bodies. However, if the bodies pass in close proximity or if we are specifically interested in the attitude dynamics of one or more of the bodies, then our analysis must be expanded to include the additional degrees of freedom associated with the distribution of mass within some or all of the bodies. Such is the case in the motion of an artificial satellite orbiting the Earth, which is ultimately our interest here. We may treat the bodies as rigid, recognizing that this is still only an approximation of the true situation. In this case, the additional degrees of freedom are due entirely to the rotation of the bodies.

We depart from the literature in that we reserve the expression *n-body problem* for the latter case, referring to the classical problem as the *n-particle problem.* The n-body problem adds an additional $3n$ degrees of freedom associated with the rotation of the rigid bodies. The equations of motion thus form a $12n$th-order system, doubling the order of the n-particle problem. Unfortunately, the number of first integrals does not do likewise. Duboshin [24] has shown that there are still ten first integrals analogous to those in the n-particle problem. Integrability is not even assured in the two-body problem, although it is for a single body. The one-body case was first treated by Euler in the mid-eighteenth century and the complete solution was eventually specified in terms of elliptic functions by Jacobi in the mid-nineteenth century. (See, *e.g.*, Whittaker [106:144].) It is often considered as a special case of the classical *heavy top problem* which treats a rigid body rotating about a fixed point in a uniform gravitational field. When the fixed point is the center of mass of the body, zero net torque results from the gravitational field. The equations of rotational dynamics are then identical to those for a single rigid body in the absence of external forces and torques. This is frequently referred to as *Euler's case.*

The problem of two bodies is the focus of the current work. We further restrict our investigation to the case where (i) one body (the *primary*) is assumed to have a total mass which is much greater than that of the other body (the *satellite*), and (ii) the mass distribution of the primary is spherically symmetric. The first condition implies the motion of the primary is essentially unaffected by the satellite and is perfectly valid for the artificial Earth satellite problem. The second condition implies the motion of the satellite is independent of the rotational dynamics of the primary and is somewhat more suspect. Clearly, the oblateness of the Earth is an issue which should be factored in at some point. However, the symmetry assumption allows considerable simplification and provides a valuable first approximation from which additional effects can be explored as perturbations at some later point. Therefore, we proceed based on the above assumptions.

We may now treat the primary as a point mass fixed at the origin in some inertial frame and consider the motion of the satellite in the corresponding central gravitational

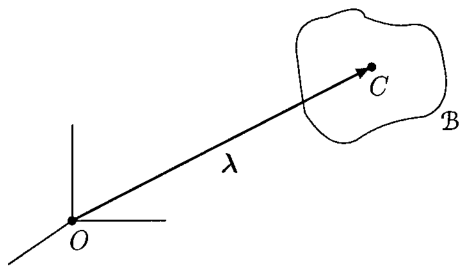


Figure 1.1 Rigid Body in a Central Gravitational Field

field. This is the *restricted two-body problem* or the *problem of a rigid body in a central gravitational field.* Figure 1.1 shows a rigid body $\mathcal{B}$ in a central gravitational field with center of attraction O which we treat as the origin of an inertial frame of reference. We characterize the motion of $\mathcal{B}$ in the inertial frame in terms of a vector $\boldsymbol{\lambda}$ representing the position of the center of mass, C, and a direction cosine matrix representing the rotation of the body (and any associated body-fixed frame) relative to the inertial frame. For particles, the restricted two-particle problem, or Kepler problem, is equivalent to the motion of a particle with reduced mass in a central gravitational field. (See, *e.g.*, Goldstein [30:70–71].) For finite bodies, however, we will show that in general no such equivalence exists.

Although this simplified system is still not integrable for an arbitrary satellite, we can find particular solutions to the equations of motion which correspond to motions of practical interest. No nontrivial equilibrium solutions exist. However, by recognizing the symmetry present in the problem and eliminating cyclic coordinates, we can reduce the order of the system. Equilibria of the reduced system do exist and are referred to as *relative equilibria.* For an arbitrary rigid body in a central gravitational field, the problem is invariant under rotations in three-dimensional space. The relative equilibria correspond to circular orbits about the primary with the body appearing stationary in an orbiting reference frame. Artificial satellites designed to remain in this relative equilibrium are referred to as *gravity-gradient satellites.* For axisymmetric bodies, the problem is also

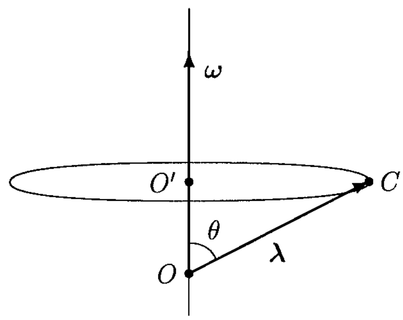


Figure 1.2 General Configuration of a Relative Equilibrium Orbit

invariant under rotations about the axis of symmetry. In this case, the relative equilibria are steady spins about the symmetry axis which is fixed in the orbiting frame. Artificial satellites designed to take advantage of a stable relative equilibrium of this nature are referred to as *spin-stabilized satellites.*

For relative equilibria of an arbitrary rigid body, the general configuration is shown in Figure 1.2 where O and C are as defined previously and O' is the orbit center. When O and O' coincide, the angular velocity vector, $\boldsymbol{\omega}$, and the position vector, $\boldsymbol{\lambda}$, are orthogonal. We refer to such a relative equilibrium as an *orthogonal relative equilibrium* and to the corresponding motion as an *orthogonal orbit.* When the two points are distinct, we refer to the relative equilibrium and its orbit as *oblique.*

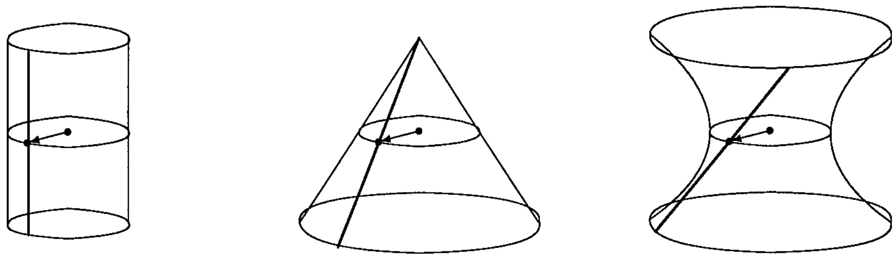
One of the difficulties of the n-body problem in contrast to the n-particle problem is that the potential involves an integral evaluated over the rigid body. In general, this integral cannot be solved in closed form. However, the integral can be represented by a series expansion in powers of the ratio of body dimension to distance from the center of attraction. For problems such as that of an artificial Earth satellite, this ratio is very small and the series converges rapidly. We are therefore justified in truncating the series expansion to approximate the potential. This approximation is then used to derive the force and torque. Alternatively, we may expand and truncate the force and torque integrals directly. This latter approach, however, raises the issue of whether or not the two series

truncations are consistent, *i.e.*, both derivable from the same truncated approximation of the potential.

The first analysis of relative equilibria for this problem was given by Lagrange[52]. He truncated the force and moment directly. Using only the dominant term in each of the series expansions, he found that the translational and rotational equations decoupled. The resulting relative equilibria for this approximate system are orthogonal and the attitude is such that one principal axis of inertia is aligned with the radial direction and another is aligned with the orbit normal. For a body with distinct moments of inertia there are twenty-four such relative equilibria and we refer to these as the *principal relative equilibria.* For this approximation, it is not possible for oblique relative equilibria to exist.

Recently Wang *et al* [104, 105] have employed a generalized (noncanonical) Hamiltonian approach to reexamine relative equilibria of arbitrary rigid satellites. This approach involves the use of phase variables that do not necessarily occur in canonical coordinate-momentum pairs. They showed that the gravitational center of attraction lies on the axis of rotation but does not always coincide with the center of the orbit. Their results are valid for the exact potential since they do not rely on a truncated series expansion. The discrepancy with the classical results arises from symmetries implicit in the truncation of the force and torque. Note that the displacement of the orbit center shown in Figure 1.2 is greatly exaggerated for clarity. The displacement is much smaller than the greatest dimension of the satellite. However, Wang *et al* [105] have demonstrated that under certain circumstances the attitude can vary significantly from the classical solution.

An advantage of the generalized Hamiltonian approach is that the formulation of the problem is the same regardless of the form of the potential (approximate or exact) and certain results can be developed which are valid for all forms. It must be emphasized that each choice of truncation order results in a different Hamiltonian system. Caution must be used when applying the results with regard to the existence and stability of relative equilibria. For most bodies with distinct inertias, the classical approximation of Lagrange is sufficiently accurate. A similar approximation exists for the noncanonical formulation



(a) Cylindrical (b) Conical (c) Hyperbolic

Figure 1.3 Relative Equilibrium Classes for Axisymmetric Satellites (after Likins [56])

of the rigid body in a central gravitational field. However, since this method uses the potential rather than the force and torque, the approximation is necessarily consistent and the translational and rotational equations remain coupled. Wang *et al* [104] have treated this approximation in part, identifying certain relative equilibria analogous to the classical relative equilibria and examining their nonlinear stability.

For axisymmetric spin-stabilized satellites, the classical treatment admits several different classes of relative equilibria. As in the classical treatment of arbitrary satellites, all orbits are orthogonal. Differences appear in the orientation of the symmetry axis. Thomson [101] and Kane *et al* [45] first considered relative equilibria with the axis of symmetry normal to the orbital plane. Pringle [81] and Likins [56] identified two additional classes of relative equilibria, one with the symmetry axis in the plane formed by the radial and orbit normal directions, and the second with the symmetry axis in the plane formed by the tangential and orbit normal directions. These relative equilibria are denoted as *cylindrical*, *conical*, and *hyperbolic*, in reference to the figure of revolution traced out by the symmetry axis during a complete orbit (see Figure 1.3). To our knowledge, no previous (canonical or noncanonical) treatment of the axisymmetric case has been accomplished which incorporates orbital-attitude coupling.

1.2 Objective and Approach

The objective of this dissertation is to apply generalized Hamiltonian methods to study the relative equilibria of a rigid body in a central gravitational field. Specifically, we wish to establish the relationships between the various approximations and more clearly define the circumstances under which a given approximation accurately represents the dynamics of a satellite in orbit about a primary.

We would like to compare the classical approximation of Lagrange with the approximations associated with the noncanonical formulation of Wang *et al.* The series truncations in the classical approximation are not consistent in the sense defined above. If the force and torque expansions are each truncated at the next higher order, the system is no longer Hamiltonian. The Hamiltonian nature of the classical approximation is purely a result of the decoupling of the translational equations from the rotational equations. (In fact, it is really two separate Hamiltonian systems.) However, an alternative viewpoint exists which allows us to treat the classical approximation as one in a series of Hamiltonian approximations: if we constrain the center of mass of the rigid body to follow a Keplerian orbit (the orbit of a particle of equal mass), then we can study the rotational dynamics associated with various approximations of the torque, one of which is the classical approximation. We can express this series of Hamiltonian systems in a noncanonical form which may then be compared to the series of noncanonical systems of Wang which represent the motion of a rigid body free to move in a central gravitational field. If we further constrain the body to move about a point fixed in inertial space, we obtain a third series of Hamiltonian systems. In this case, the effects of centripetal acceleration due to orbital motion about the center of attraction are eliminated, but central field effects remain.

Thus, we have identified three related problems of rigid body motion in a central gravitational field: motion about a fixed point, motion about a point on a Keplerian orbit, and free motion. From these, we develop a heirarchy of Hamiltonian systems which are related to the two-body problem. The Hamiltonian system for each problem may be expressed in a noncanonical form. These three problems of rigid-body motion form

a dimension of our hierarchy reflecting the constraints on the body. Since each of the noncanonical formulations allows any order of approximation of the potential, the degree of approximation forms a second dimension of the hierarchy. A third dimension is associated with the degree of symmetry present in the rigid body: spherical, axial, or none. It should be noted that the degree of approximation determines the stringency of the symmetry requirements. For instance, the second-order approximation reduces the axial symmetry requirement to one of *dynamic symmetry*, *i.e.*, equal moments of inertia.

The noncanonical formulations of the Keplerian and fixed-point problems are a new result. The fixed-point problem is a generalization of the heavy-top system treated by Maddocks [62] and is not treated further. From the systems associated with the Keplerian and free rigid-body problems in the hierarchy, we identify three particular Hamiltonian systems for further investigation: the second-order approximation of the Keplerian system (the noncanonical formulation of the classical approximation), the second-order approximation of the free rigid-body system, and the exact free rigid-body system. These systems represent increasingly accurate approximations of the two-body system. From the noncanonical formulation of each, we identify the relative equilibria and investigate their stability (linear and nonlinear). We treat both the arbitrary and the axisymmetric cases. In particular, we are interested in extending the classical results to the more rigorous setting of coupled translational and rotational dynamics.

The analysis presented here enables further understanding of all three systems. Although the results for the Keplerian problem are well known (at least for the second-order approximation), the noncanonical derivation has several advantages. One such advantage is that rotations are not parameterized in terms of Euler angles so that, for instance, the relative equilibrium conditions for the axisymmetric case in the Keplerian problem are stated in a more general form than has previously been given. While the case of an arbitrary body for the free rigid-body problem has been previously explored by Wang *et al*, we extend their results with regard to the second-order approximation and the exact

potential. The results for the axisymmetric case in the free rigid-body problem are entirely new.

1.3 Overview

The presentation of this dissertation consists of two primary parts. In the first part, we lay out the background material and develop the hierarchy of systems relating to the two-body problem. In the second part, we narrow our focus to three specific systems and present the detailed analysis of their relative equilibria for comparison.

The first part consists of Chapters 2–5. In Chapter 2, we review the literature on the three related problems in mechanics identified above which deal with the motion of a rigid body in a central gravitational field. Within the astronautics literature, we emphasize the work pertaining to gravity-gradient and spin-stabilized satellites. In Chapter 3, we discuss relative equilibria of Hamiltonian systems. Relevant definitions and characterizations of relative equilibria are presented. A review of stability definitions and analysis techniques is given with emphasis on those concepts which result directly from the Hamiltonian framework. Chapter 4 and Chapter 5 examine the two-body problem and develop the heirarchy of systems which approximate it in the presence of a spherical primary. We develop the relationships between the various systems and identify the assumptions implicit as we move further down the heirarchy.

The second part, consisting of Chapters 6 through 8, investigates three particular Hamiltonian systems which progressively improve upon the approximation of the two-body problem. In Chapter 6, we examine the second-order approximation of the Keplerian problem. This system is equivalent to the classical approximation which appears throughout the literature, but we take advantage of the noncanonical formulation to state the relative equilibrium condition in a more general form. Chapter 7 presents the second-order approximation of the free-body problem. Here we introduce orbital-attitude coupling while maintaining the simple torque expression of the classical approximation. Finally, in Chapter 8 we address the exact form of the free-body problem, which further adds the gravita-

tional torque effects due to third- and higher-order inertia integrals. In each chapter, we develop the noncanonical Hamiltonian system in nondimensional form and determine the conditions for existence of relative equilibria. In Chapters 6 and 7, we identify all relative equilibria for both the arbitrary and axisymmetric cases and investigate their stability. In Chapter 8, we build on the work of Wang *et al* [105] to explore the conditions for existence of orthogonal and oblique relative equilibria. Some stability results are also presented.

Finally, in Chapter 9 we summarize our results and identify possible directions for further investigation. The appendices include a presentation of some of the fundamentals of generalized Hamiltonian mechanics, a development of the gravitational potential (including series expansions used to approximate the potential), and proof of an identity used to derive the noncanonical formulation of the Keplerian problem.

II. Literature Review

In this chapter, we review much of the previous work on rigid-body dynamics in a central gravitational field. While our analysis of the dynamics will be partitioned to treat three separate problems (motion about a fixed point, motion about a point on a Keplerian orbit, and free motion), no such distinction can be made in a review of the literature with regard to the latter two problems. We therefore treat them as a single problem which we refer to as the orbiting satellite problem. We begin with a review of the classical origins of these problems. We then examine modern developments of the orbiting satellite problem. Obviously, a great deal of this work comes from the field of astronautics. We place particular emphasis on those works directly related to gravity-gradient and spin-stabilized satellites. Most of this work has been extended to more complex multibody systems such as dual-spin satellites for which we merely provide references to recent literature reviews. We then review previous work on the fixed point problem. This work has appeared almost exclusively within the Russian literature and, due to the more abstract nature of the problem, has been presented in the more general setting of the field of applied mechanics. We close with a discussion of some textbook treatments of these problems which the reader may find useful.

2.1 Classical Origins

The motion of a rigid body in a central gravitational field is a simplification of the two-body problem. As such, it is integrally tied to celestial mechanics. Much of the early work in the two-body problem was directed towards identifying trajectories or orbits of heavenly bodies. It is well known that in 1609 Kepler [48] first identified the planetary orbits as ellipses with the sun at one focus and in 1687 Newton published his *Principia* [78] showing the corresponding force of attraction is inversely proportional to the square of the distance between the bodies. Newton also showed that a spherically symmetric attracting body results in a force equivalent to that of a point mass located at the center of mass of

the attracting body. This provided a solid mathematical foundation for the study of orbits of celestial bodies.

The attitude motion of heavenly bodies has received less attention historically. Two notable exceptions are the Earth and the Moon — the former because its attitude directly influences the observations of other celestial bodies and the latter because its close proximity to the Earth has significant effects and also makes attitude motions more readily observable. In 1749, while attempting to explain the precession of the equinoxes, d'Alembert [19] and Euler [27] developed the equations of motion for a rigid body in a central gravitational field, first deducing the presence of a gravitational torque on a rigid body of arbitrary shape. In 1780, interest in the librations of the Moon led Lagrange [52] to study the equilibria of these equations expressed in an orbiting reference frame. He identified twenty-four such relative equilibria corresponding to each of the possible combinations of principal inertia axes aligned with orbital frame axes. He developed sufficient conditions for stability in terms of the inertia ratios.

Note that these two cases typify in certain respects the types of motion we are interested in here. The Moon is a tri-inertial body and its motion in orbit about the Earth is a libration about a stable relative equilibrium for an arbitrary body. In contrast, the Earth is approximately dynamically symmetric and its motion in orbit about the Sun is essentially a relative equilibrium for an axisymmetric body. In the case of the Earth, however, factors such as the presence of the Moon introduce other significant effects.

According to Roberson [92], a considerable body of work on this problem exists from the nineteenth century, however, advances were limited due to the narrow scope of applications under consideration. The development of the artificial satellite brought renewed interest.

2.2 Modern Developments

2.2.1 The Orbiting Satellite Problem.

Equations of Motion and Relative Equilibria. The moment equations for a rigid body in a central gravitational field had become a part of the astronomical literature by

the turn of the century. (See, *e.g.*, Tisserand [103] or Routh [93].) However, at the start of the space age, they went through a series of "rederivations." As Roberson [85] points out, some of these new derivations were useful for presenting the equations in modern vectorial form or in a more general setting (generalized potentials, nonspherical primaries, *etc.*).

In 1956, Roberson and Tatistcheff [88] developed the potential function for a finite rigid body in the gravitational field of a homogeneous oblate spheroid. For the special case of a spherical primary, their approximation is equivalent to MacCullagh's [58] approximation for the potential of a body evaluated at a distant point. The difference is just a reversal of the roles of primary and satellite.[1] This approximation neglects terms in the series expansion of the potential which involve third-order or higher inertia integrals and is justified under the assumption that the size of the satellite is very small in comparison to the distance from the center of the primary. This same approximation has continued to be used in various forms throughout nearly all the literature, usually with the additional caveat that only the dominant term in the resulting force is kept. This is the same approximation used by Lagrange which we refer to as the classical approximation. In 1958, Roberson [83, 89] outlined the problem of satellite attitude control and developed expressions for several torques, including the gravity-gradient torque. He identified the stability or instability of various configurations in terms of principal inertia ratios, listing the configurations previously found by Lagrange as the only stable configurations.

In 1958, Duboshin [24] derived the translational and rotational equations of motion for a system of n rigid bodies of finite size. He demonstrated that in general the problem has ten first integrals, analogous to the first integrals for the motion of n point masses acting under their mutual gravitation. Also, Beletskii [12] considered the effects of aerodynamic and gravitational torques along with orbital regression on satellite attitude motion. A year later, Beletskii [13] restricted his analysis to admit only the gravitational torque.

[1]The current usage should be contrasted with the more typical usage of MacCullagh's approximation where the satellite is a point mass and we are interested in the effects of the nonspherical mass distribution of the primary on the orbit of the satellite.

He demonstrated the nonlinear stability of Lagrange's stable equilibria and examined librations about those equilibria. In an independent effort in 1961, DeBra and Delp [23], working from the derivations of Roberson [83], applied a linear stability analysis to the problem. They demonstrated the weaker condition of spectral stability for Lagrange's stable equilibria as well as the existence of a second set of inertia configurations which are spectrally stable. Beletskii [13] had also identified this second set of configurations but had incorrectly declared them unstable when considered in the nonlinear problem.[2] In 1963, Garber [29] extended the results of DeBra and Delp [23] to consider aerodynamic torques. He determined the presence of an instability under the influence of a body-fixed torque.

In 1963, Michelson [72, 73] presented another derivation of the equations of motion for a rigid body in a central gravitational field. He claimed the existence of a one-parameter family of relative equilibria. This claim was refuted by Kane and Likins [43], who argued that the moment equations linearized about the known equilibria do not permit a family of equilibria. As a direct result of this debate, Likins and Roberson [55] gave a proof of the uniqueness of the twenty-four relative equilibria for a tri-inertial rigid body. They converted the equilibrium conditions to an eigenvalue problem and showed that the radial and binormal unit vectors are eigenvectors of the inertia matrix. In 1968, however, Meirovitch [69] pointed out that this proof relied on the truncated approximation of the potential and that for inertially symmetric or nearly symmetric rigid bodies, higher-order terms in the series expansion of the potential should be included.

In 1968, Roberson [90, 91] considered the possibility that orbits of relative equilibria could lie in a plane which does not contain the center of force. Roberson was specifically interested in the motion of a gyrostat and ruled out the possibility of these oblique orbits for a simple rigid body. However, his findings were a direct result of the truncated form of the potential and in 1985 Barkin [9] showed the existence of oblique orbits by including third-order inertia integrals in the potential. In 1991, Wang, Krishnaprasad, and Maddocks [104]

[2]Meyer and Hall [71] point out that during this period several proofs of instability appeared in the literature which were incorrect (*e.g.*, Malkin [64] and La Salle and Lefschetz [51]).

developed an approach to solution of the exact problem using a noncanonical Hamiltonian formulation and variational principles. Their method agreed with previous results when applied to approximations of the Hamiltonian corresponding to truncated forms of the potential. In 1992, Wang, Maddocks, and Krishnaprasad [105] showed that the classical relative equilibria are guaranteed to exist if the rigid body has three mutually orthogonal planes of symmetry. This was implicitly assumed in previous developments which truncated the potential at second order. They examined the exact problem for a specific rigid body and showed numerically the existence of oblique orbits. They pointed out that while the displacements of the orbit plane from the center of attraction were necessarily small, the equilibrium attitudes associated with these orbits could vary greatly from those of the classical relative equilibria. In 1993, Provost [82] attempted to verify these findings and further examine the nature of these equilibria. Unfortunately, this effort met with limited success due apparently to difficulties associated with the ill-conditioned numerics in the problem.

Librations about the Relative Equilibria. In 1957, Klemperer and Baker [49] began to examine the problem of satellite librations about a relative equilibrium. They developed equations for planar librations of both a dumbbell-shaped satellite and a prolate spheroid. They examined the frequency of the librations and suggested the possibility of using instruments to sense these librations for satellite control. Davis [20] also examined the planar librations of a dumbbell-shaped satellite about its stable relative equilibria. He developed linearized equations for roll, pitch, and yaw. Baker [8] and Schindler [97] were among those extending these analyses to examine planar librations for elliptical orbits in the early 1960's.

As mentioned above, Beletskii [13] also considered the libration problem. It is noteworthy that he distinguished between the *restricted problem* where a Keplerian orbit is assumed and the resulting attitude motion treated independently and the *unrestricted problem* where the orbital and attitude motion are coupled. He derived the general li-

bration equations for the restricted problem in a circular orbit and the planar libration equations for the restricted problem in an elliptical orbit. He determined an expression for the period of the librations. The general libration equations for an axisymmetric rigid body in a circular orbit were also derived by Auelmann [7] in 1963. DeBra and Delp [23] and Michelson [73] also discussed the frequency of librations for circular orbits in their work mentioned previously. In 1965, Kane [41] reexamined the analysis of DeBra and Delp. By considering the amplitude of librations, he showed that while the relative equilibria found by Lagrange and DeBra and Delp are stable, librations about some of these configurations can be unstable for amplitudes as small as one degree.

In the late 1960's, Brereton and Modi [74, 17, 75, 76] employed both numerical and analytical methods to study the librations in circular and elliptical orbits. They found periodic solutions with periods of several times the orbital period. They also numerically determined the existence of invariant surfaces. Marandi and Modi [65] applied Morse theory to determine the libration bounds of asymmetric satellites. More recently there has been considerable interest in chaotic motions associated with the pitch dynamics in an elliptical orbit (*e.g.*, [47, 31]).

Several symposia have been dedicated to the subject of gravity-gradient satellites (*e.g.*, [77, 95]). These proceedings provide a source of theoretical and applied works. Provost [82] has also recently reviewed the gravity-gradient satellite literature.

Spin-Stabilized Satellites. In 1962, Thomson [101] and Kane, Marsh, and Wilson [45] showed that for axisymmetric satellites it is possible to stabilize the direction of the symmetry axis normal to the orbit plane by spinning the body. This is true for both oblate satellites which are naturally stabilized by the gravity gradient and prolate satellites which are not. Kane and Shippy [42] extended this analysis to consider spinning asymmetric rigid bodies. Pringle [81] and Likins [56] demonstrated the possibility of directional stability for axisymmetric satellites along directions other than the orbit normal and examined the bounds for librations of the symmetry axis. Meirovitch and Wallace [68]

extended these results to consider aerodynamic torque effects. With the development of the dual-spin concept, this research was quickly overtaken by interest in more complex multibody systems.

Gyrostats and Dual-Spin Satellites. Roberson [84] first considered the influence of rotors on the stability of a rigid body in a central gravitational field in 1958. A large body of literature exists on this area starting in the mid 1960's. It continues to be a topic of interest, however, it is not directly relevant to this dissertation. We refer the reader to [94, 85, 86, 92, 87] for discussions of developments relating to gyrostats and dual-spin satellites and to Hall [32] for a recent survey of the literature relating to dual-spin satellites.

2.2.2 The Fixed-Point Problem. The problem of rigid-body motion about a fixed point in a central gravitational field may be viewed as a simplification of the orbiting satellite problem in which we eliminate the rotational effects due to centripetal acceleration and consider only the gravitational torques. In this case, we specify the fixed point to be the center of mass. Alternatively, it may be viewed as a generalization of the classical heavy top problem. Here the fixed point is not specified and the classical problem arises as one of the approximations based on truncation of a series expansion of the potential. There is a large body of literature on the heavy top problem. (See, *e.g.*, Whittaker [106] or Leimanis [53].) Maddocks [62] and Lewis *et al* [54] have recently reexamined this classical problem using generalized Hamiltonian methods.

Several integrable cases of the classical problem are well known. Two of these — the cases of Euler and Lagrange — have analogous integrable cases for the problem in a central gravitational field. For the case analogous to Euler's, an additional first integral was discovered by de Brun [22] and the problem was solved by Kobb [50] and Harlamova [33]. The case analogous to Lagrange's was reduced to quadratures by Beletskii [11, 14]. An additional integrable case exists when the body has complete dynamic symmetry (three equal inertias) with respect to axes originating at the fixed point. The case of Euler and the case of complete dynamic symmetry are only integrable for the classical potential

approximation. The case of Lagrange is integrable for all approximations. Arkhangelskii [2, 3, 5, 4] has shown that these are the only integrable cases for the classical approximation.

In more recent work, Bogoyavlenskii [16] has generalized the case of Euler to show that the classical approximation for an arbitrary (non-spherical) primary is integrable. His formulation of the problem is noncanonical. Sulikashvili [99, 100] has examined the relative equilibria of bodies which admit regular polyhedron symmetry groups and are fixed at their center of mass. Periodic solutions of the fixed point problem have been investigated by El-Sabaa [26].

2.3 Textbook Treatments

Most celestial mechanics texts treat only the n-particle problem. However, the work by Duboshin on the n-body problem appears as a chapter in his own text [25] with the addition of a section in which the equations of motion are converted from Lagrangian to Hamiltonian form. For the problem of a rigid body in a central gravitational field, the text by Beletskii [15], written in 1965, is remarkably complete and remains one of the most thorough treatments. Appendix 1 covers the motion about a fixed point, while the main text examines both the restricted (Keplerian) and unrestricted (free body) cases. In the same year, Leimanis [53] presented a survey of the literature which reviews some of the material found in [15] but includes the heavy top problem as well as some analysis of the dynamics of gyrostats and gyroscopes. A more recent presentation of satellite attitude dynamics is given by Hughes [40]. This treatise develops the dynamics from first principles and concludes with three full chapters devoted to gravity-gradient satellites, spin-stabilized satellites, and gyrostats and dual-spin satellites. The presentation includes most of the pertinent results from the above literature as well as application-oriented discussions. The text by Kane, Likins, and Levinson [44] presents much of the dynamics, but in a less readable fashion. Elementary presentations of satellite attitude dynamics can also be found in the texts by Kaplan [46], Thomson [102], and Wiesel [107]. Meirovitch [70] includes a chapter on celestial mechanics (including the n-particle problem and Earth

attitude dynamics) and a chapter on spacecraft dynamics (including relative equilibria of gravity-gradient and spin-stabilized satellites) as applications in a text on Lagrangian and Hamiltonian mechanics.

III. Stability of Relative Equilibria of Hamiltonian Systems

In this chapter, we examine relative equilibria which arise as critical points of noncanonical Hamiltonian systems and explore their stability. Our objective is to provide a collection of analytical tools to be used in later chapters when we investigate particular Hamiltonian systems of interest. We presume the reader has some familiarity with properties of Hamiltonian systems. A primer on both canonical and noncanonical systems is provided in Appendix A.

We begin by presenting a definition of relative equilibrium as well as definitions of several different types of stability. Relative equilibria are then characterized in terms of a variational principal. Subsequently, we focus on two types of stability — spectral stability and nonlinear stability — which are considered in some detail.

A comprehensive review of the existing knowledge on stability of Hamiltonian systems would go far beyond the scope of the current effort. We focus on topics directly applicable to the systems considered in later chapters and apply methods recently developed specifically for treatment of noncanonical Hamiltonian systems. However, we emphasize spectral stability to a greater degree than most of the current literature (with the exception of Howard and Mackay [36] and Howard [38, 39]). Therefore, the section on spectral stability contains some results which have not appeared elsewhere in the literature to our knowledge. Our reasoning for increased emphasis on spectral stability is to present a more complete picture of stability in terms of the parameters which specify the configuration of the dynamical system. This argument is presented more fully in the section on stability. For nonlinear stability, we discuss several methods which exploit the advantages of noncanonical Hamiltonian formulations — two methods which consider the relative equilibrium as an unconstrained extremum and two which treat it as a constrained extremum.

3.1 Definitions

When we refer to the stability of a dynamical system, we are describing the tendency for the trajectory of the system to remain close to a given reference trajectory. Exactly

in what sense the trajectories are close must be more clearly defined. We consider the stability of trajectories which correspond to a particular type of solution to the equations of motion:

Definition 3.1 (Equilibrium, Relative Equilibrium). For an autonomous dynamical system

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}), \tag{3.1}$$

a point $\mathbf{z}_e$ which satisfies $\mathbf{f}(\mathbf{z}_e) = 0$ is an *equilibrium.* If the dynamical system (3.1) describes motion relative to a moving coordinate system, then we refer to an equilibrium of this system as a *relative equilibrium.*

Along with the dynamical system (3.1), we also consider its linearization. A Taylor series expansion of (3.1) about the equilibrium $\mathbf{z}_e$ gives

$$\dot{\delta\mathbf{z}} = \mathbf{A}(\mathbf{z}_e)\delta\mathbf{z} + \mathcal{O}\left(\delta\mathbf{z}^2\right) \tag{3.2}$$

where $\delta\mathbf{z} = \mathbf{z} - \mathbf{z}_e$ is the perturbation vector and $\mathbf{A}(\mathbf{z})$ is the Jacobian matrix

$$\mathbf{A}(\mathbf{z}) = \frac{d\mathbf{f}(\mathbf{z})}{d\mathbf{z}}. \tag{3.3}$$

Truncating all terms above first order in Equation (3.2) gives the linearized system and we consider stability of the origin, $\delta\mathbf{z} = \mathbf{0}$.

Four definitions of stability which often appear in the literature[1] are:

Definition 3.2 (Nonlinear Stability). An equilibrium $\mathbf{z}_e$ is *nonlinearly stable* (or *Liapunov stable*) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $||\mathbf{z}(0) - \mathbf{z}_e|| < \delta$, then $||\mathbf{z}(t) - \mathbf{z}_e|| < \varepsilon$ for $t > 0$.

Definition 3.3 (Formal Stability). An equilibrium $\mathbf{z}_e$ is *formally stable* if a conserved quantity is identified for which the first variation is zero and the second variation is positive or negative definite when evaluated at the equilibrium.

Definition 3.4 (Linear Stability). An equilibrium $\mathbf{z}_e$ is *linearly stable* (or *infinitesimally stable*) if the linearized system is nonlinearly stable.

[1]See, *e.g.*, Holm *et al* [35].

Definition 3.5 (Spectral Stability). An equilibrium $\mathbf{z}_e$ is *spectrally stable* if the spectrum of the linearized operator $\mathbf{A}(\mathbf{z}_e)$ has no positive real part.

We also must consider the converse:

Definition 3.6 (Instability). An equilibrium $\mathbf{z}_e$ is *unstable* if it is not nonlinearly stable.

In general, we have [35:4–5]

formal stability $\Rightarrow$ linear stability $\Rightarrow$ spectral stability

where $A \Rightarrow B$ indicates "A implies B". The former implication follows since the quadratic form associated with the second variation of the conserved quantity is the square of a norm which may be used to prove linear stability. The latter implication is a result of the fact that for an unstable eigenspace no sufficiently small perturbation can be found (not spectrally stable $\Rightarrow$ not linearly stable). For the finite-dimensional systems we consider here, we also have

formal stability $\Rightarrow$ nonlinear stability.

This is a classical result due to Liapunov [57:62] and is a generalization of a theorem attributed to Lagrange and Dirichlet[2] which states that an equilibrium which occurs at an isolated minimum of the potential function is nonlinearly stable.

3.2 Characterization of Relative Equilibria

We are interested in the class of dynamical systems known as Hamiltonian systems for which the equations of motion (3.1) take the special form

$$\dot{\mathbf{z}} = \mathbf{J}(\mathbf{z})\nabla H(\mathbf{z}) \tag{3.4}$$

[2]See, *e.g.*, Marsden and Ratiu [67:31].

as described in Appendix A. For these systems, a relative equilibrium $\mathbf{z}_e$ must satisfy the condition

$$\mathbf{J}(\mathbf{z}_e)\nabla H(\mathbf{z}_e) = \mathbf{0}. \tag{3.5}$$

One interpretation of this equation is that $\mathbf{z}_e$ is a relative equilibrium if and only if the gradient of the Hamiltonian lies in the null space of the structure matrix $\mathbf{J}(\mathbf{z}_e)$. If the structure matrix is singular, then there exist functions known as Casimir functions which are conserved (independent of the Hamiltonian). The gradient of each of these functions lies in the null space of $\mathbf{J}(\mathbf{z})$ for any choice of $\mathbf{z}$. If the system has order n and the structure matrix has rank $n-m$ then we may identify m Casimir functions such that their gradients are linearly independent and form a basis for the null space of $\mathbf{J}(\mathbf{z})$. Thus, at relative equilibrium the gradient of the Hamiltonian must equal some linear combination of the gradients of the Casimir functions:

$$\nabla H(\mathbf{z}_e) = \sum_{i=1}^{m} \mu_i \nabla C_i(\mathbf{z}_e). \tag{3.6}$$

Whereas Condition (3.5) is a system of n algebraic equations in n unknowns (the phase variables at equilibrium), Condition (3.6) is a system of n equations in $n+m$ unknowns (the phase variables and the multipliers, μ_i). We additionally require the m equations

$$C_i(\mathbf{z}) = k_i \qquad (i = 1, 2, \ldots, m) \tag{3.7}$$

where the k_i may be constants or parameters of the physical system. We shall see examples of each possibility in later chapters. Conditions (3.6) and (3.7) are equivalent to Condition (3.5).

Following Maddocks [62], we introduce the function

$$F(\mathbf{z}) = H(\mathbf{z}) - \sum_{i=1}^{m} \mu_i C_i(\mathbf{z}). \tag{3.8}$$

Critical points of this function are relative equilibria since the condition

$$\nabla F(\mathbf{z}_e) = \mathbf{0} \tag{3.9}$$

is equivalent to Condition (3.6). It is immediately apparent that Conditions (3.9) and (3.7) arise from the constrained variational principle

$$\text{make stationary } H(\mathbf{z}) \text{ subject to } C_i(\mathbf{z}) = k_i. \tag{3.10}$$

Here the μ_i serve as the Lagrange multipliers and the function $F(\mathbf{z})$ is the variational Lagrangian.[3] In many respects, F assumes the role which the Hamiltonian serves for canonical systems. This is not too surprising since canonical systems have a nonsingular structure matrix (so that there are no constraints) and the variational Lagrangian reduces to $F(\mathbf{z}) = H(\mathbf{z})$. This variational characterization not only provides a direct method of determining the conditions for relative equilibrium, it will also prove useful in both the spectral and nonlinear stability analyses.

3.3 Stability of Relative Equilibria

Our intent is to examine the nonlinear stability of systems parameterized by physical quantities. Nonlinear stability criteria (which typically arise from proofs of formal stability) tend to provide sufficient rather than necessary conditions for nonlinear stability. They therefore divide the parameter space into two types of regions: nonlinearly stable configurations and configurations of unknown stability. This is shown conceptually in Figure 3.1(a) where the λ_i represent physical parameters.

We would like to present a more complete picture of stability in terms of the parameters which specify the configuration. From the definitions above, it can be deduced that an equilibrium is linearly stable if and only if it is spectrally stable and the Jordan block cor-

[3] We use the adjective *variational* to clearly distinguish this function from the Lagrangian function which arises in mechanics.

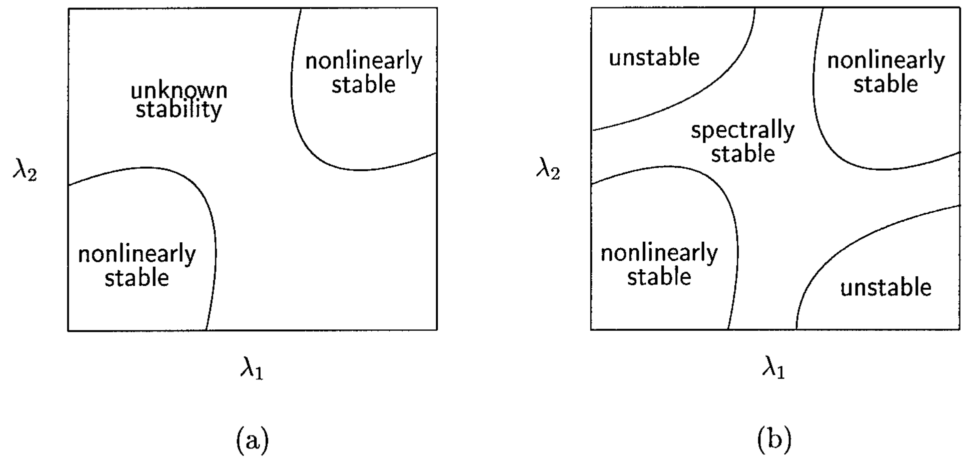


Figure 3.1 Stability Regions with (a) Nonlinear Stability Analysis and (b) Nonlinear and Spectral Stability Analyses

responding to each imaginary eigenvalue is one dimensional [34]. Furthermore, the spectral and linear stability boundaries in parameter space are identical for Hamiltonian systems. We opt to consider only spectral stability and avoid the concerns of repeated roots. It turns out these borderline cases will often correspond to configurations with additional symmetry which we treat separately. By considering spectral stability in addition to nonlinear stability, we can refine our categorization of parameter space into three types of regions: nonlinearly (and spectrally) stable regions, spectrally (and nonlinearly) unstable regions, and spectrally stable regions. This is shown conceptually in Figure 3.1(b). The effort required to analyze spectral stability can be considerably less than that required to analyze nonlinear stability and it allows us to focus the nonlinear analysis on only those regions which are spectrally stable.

3.3.1 Spectral Stability.

Linearized Equations of Motion. For Hamiltonian systems, we can develop an expression for the linear system matrix $\mathbf{A}(\mathbf{z}_e)$ in terms of the structure matrix and the variational Lagrangian [62]. Linearization of Equation (3.4) about relative equilibrium

gives

$$\dot{\delta\mathbf{z}} = \left[\nabla\mathbf{J}(\mathbf{z}_e)\nabla H(\mathbf{z}_e) + \mathbf{J}(\mathbf{z}_e)\nabla^2 H(\mathbf{z}_e)\right]\delta\mathbf{z}. \tag{3.11}$$

However, for each Casimir function we have

$$\mathbf{J}(\mathbf{z})\nabla C_i(\mathbf{z}) = \mathbf{0} \tag{3.12}$$

and differentiation with respect to the phase variable vector gives

$$\nabla\mathbf{J}(\mathbf{z})\nabla C_i(\mathbf{z}) + \mathbf{J}(\mathbf{z})\nabla^2 C_i(\mathbf{z}) = \mathbf{0}. \tag{3.13}$$

Combining this result with Equation (3.6), we may then conclude

$$\nabla\mathbf{J}(\mathbf{z}_e)\nabla H(\mathbf{z}_e) = -\sum_{i=1}^{m}\mu_i\mathbf{J}(\mathbf{z}_e)\nabla^2 C_i(\mathbf{z}_e) \tag{3.14}$$

which may be substituted into Equation (3.11) to give

$$\dot{\delta\mathbf{z}} = \mathbf{J}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\delta\mathbf{z}. \tag{3.15}$$

Thus, given any Hamiltonian system, we may write the system matrix for the linearized equations of motions directly (after determining suitable Casimir functions) as

$$\mathbf{A}(\mathbf{z}_e) = \mathbf{J}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e) \tag{3.16}$$

without performing linearization.

Eigenstructure of Hamiltonian Systems. The eigenvalues of the linear system matrix are the roots of the characteristic equation

$$P(s) = \det\left[s\mathbf{1} - \mathbf{A}(\mathbf{z}_e)\right] = 0. \tag{3.17}$$

For low-order systems, we may solve for the eigenvalues explicitly. More typically, we apply an algorithm such as the Routh-Hurwitz method to determine conditions on the coefficients of $P(s)$ which are necessary and sufficient to guarantee no eigenvalues lie in the right half-plane. For a general high-order polynomial, development of these conditions can be a daunting task. Fortunately, Hamiltonian systems have special properties with regard to both the form of the characteristic polynomial and the eigenstructure which greatly simplify this effort.

Property 1. *Eigenvalues of the linear system matrix are symmetric about both the real and imaginary axes.*

This is a well-known property for canonical Hamiltonian systems. Symmetry about the real axis is a property of all real matrices. Symmetry about the imaginary axis results from $\mathbf{A}(\mathbf{z}_e)$ being the product of a skew-symmetric and a symmetric matrix.

Property 2. *A zero eigenvalue exists for each linearly independent Casimir function.*

The gradient of a Casimir function lies in both the left and right nullspaces of the structure matrix $\mathbf{J}(\mathbf{z}_e)$. Therefore, it also lies in the left nullspace of $\mathbf{A}(\mathbf{z}_e) = \mathbf{J}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)$. In other words, it is the left eigenvector associated with a zero eigenvalue. The linear independence of the Casimir function gradient vectors assures us that the multiplicity of the zero eigenvalue must be at least as great as the number of independent Casimir functions.

Property 3. *An additional pair of zero eigenvalues exists for each first integral which is associated with a symmetry of the Hamiltonian, and which is linear in the phase variables.*

For any first integral, we have

$$\dot{C}(\mathbf{z}_e) = \nabla C(\mathbf{z}_e) \cdot \mathbf{J}(\mathbf{z}_e)\nabla H(\mathbf{z}_e) = 0. \tag{3.18}$$

Differentiating with respect to the phase variable vector gives

$$\nabla^2 C(\mathbf{z}_e) \cdot \mathbf{J}(\mathbf{z}_e)\nabla H(\mathbf{z}_e) + \nabla C(\mathbf{z}_e) \cdot \nabla \mathbf{J}(\mathbf{z}_e)\nabla H(\mathbf{z}_e) + \nabla C(\mathbf{z}_e) \cdot \mathbf{J}(\mathbf{z}_e)\nabla^2 H(\mathbf{z}_e) = \mathbf{0}. \tag{3.19}$$

The first term is zero because $C(\mathbf{z}_e)$ is linear in the phase variables. By Equations (3.11)–(3.16), the rest reduces to

$$\nabla C(\mathbf{z}_e) \cdot \mathbf{A}(\mathbf{z}_e) = \mathbf{0}. \tag{3.20}$$

Thus, the gradient of such a first integral is a left eigenvector of the linear system matrix associated with a zero eigenvalue. If the gradient of this integral and the gradients of the Casimir functions are linearly independent, then there will be an additional zero eigenvalue. However, for an nth-order system with m independent Casimir functions, the $n - m$ eigenvalues associated with the $(n - m)/2$ degrees of freedom must occur in pairs or quadruplets symmetric about the real and imaginary axes.[4] Therefore, the symmetry must drive a pair (and possibly a quadruplet) to the origin.

Consideration of these properties leads us to conclude that the characteristic polynomial for a Hamiltonian system is of the form

$$P(s) = s^m(s^{n-m} + A_{n-m-2}s^{n-m-2} + A_{n-m-4}s^{n-m-4} + \cdots + A_2 s^2 + A_0). \tag{3.21}$$

The even form of the polynomial inside the parentheses is a byproduct of the first property and allows the polynomial to be treated as a polynomial in $\sigma = -s^2$ of order $(n - m)/2$. For an additional integral of motion associated with a symmetry of the Hamiltonian as described in the third property, A_0 would go to zero so that an additional s^2 would factor out. While these properties lead to considerable simplification of the characteristic polynomial, we can simplify still further in many cases.

[4]In Appendix A, we discuss why $n - m$ must be even.

Decoupling of Linear Subsystems. For noncanonical systems, there is no pairing of phase variables into conjugate pairs. Hence, we are completely free to rearrange the phase variables as necessary to simplify our computations. We will often find that the linearized system decouples into (1) entirely independent subsystems so that the linear system matrix may be block-diagonalized, or (2) subsystems for which the coupling is only in one direction so that the linear system matrix may be block-triangularized. For instance, suppose we have a Hamiltonian system with phase variable vector $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5)$ for which the linear system matrix takes the form

$$\mathbf{A}(\mathbf{z}_e) = \begin{bmatrix} a_{11} & 0 & a_{13} & a_{14} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & 0 & a_{33} & a_{34} & 0 \\ a_{41} & 0 & a_{43} & a_{44} & 0 \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}. \tag{3.22}$$

We may consider the Hamiltonian system with permuted phase variable vector

$$\tilde{\mathbf{z}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_3 \\ z_4 \\ z_2 \\ z_5 \end{pmatrix} \tag{3.23}$$

for which the linear system matrix would become

$$\tilde{\mathbf{A}}(\tilde{\mathbf{z}}) = \left[\begin{array}{ccc|cc} a_{11} & a_{13} & a_{14} & 0 & 0 \\ a_{31} & a_{33} & a_{34} & 0 & 0 \\ a_{41} & a_{43} & a_{44} & 0 & 0 \\ \hline a_{21} & a_{23} & a_{24} & a_{22} & a_{25} \\ a_{51} & a_{53} & a_{54} & a_{52} & a_{55} \end{array}\right]. \tag{3.24}$$

Recognition of this decoupling into block-diagonal or block-triangular form greatly simplifies the computation of the characteristic equation since, in either case, we may then treat each subsystem independently. That is, the characteristic polynomial is

$$P(s) = |s\mathbf{1} - \mathbf{A}(\mathbf{z}_e)| = \begin{vmatrix} s\mathbf{1} - \mathbf{A}_{11}(\mathbf{z}_e) & \mathbf{0} \\ -\mathbf{A}_{21} & s\mathbf{1} - \mathbf{A}_{22}(\mathbf{z}_e) \end{vmatrix} = |s\mathbf{1} - \mathbf{A}_{11}(\mathbf{z}_e)|\,|s\mathbf{1} - \mathbf{A}_{22}(\mathbf{z}_e)| \tag{3.25}$$

where the $\mathbf{A}_{ij}(\mathbf{z}_e)$ are the submatrices of the block-triangular (or block-diagonal) form. We can easily extend this to treat more than two independent subsystems through repeated application of the above procedure.

Stability Criteria. The above results guarantee that for an nth-order Hamiltonian system with m independent Casimir functions, we will only need to analyze a polynomial of order $n - m$ or less in s^2 (or $(n-m)/2$ in $\sigma = -s^2$). By the symmetry property of the roots, the system can only be spectrally stable when all the eigenvalues lie on the imaginary axis. We thus need to determine under what conditions this requirement is satisfied. Two methods are available for this analysis. One is a special case of Routh's method and the other is based on Sturm sequences. We briefly describe both methods here and refer the reader to more detailed descriptions.

Routh's method is commonly treated in textbooks on linear systems or linear control theory. It involves the construction of a table based upon the coefficients of the characteristic polynomial. For an nth-order polynomial

$$a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \tag{3.26}$$

the table has n rows. The first two rows of the table are given by (let n be even for this example)

$$\begin{array}{c|ccccc} s^n & a_n & a_{n-2} & a_{n-4} & \cdots & a_0 \\ s^{n-1} & a_{n-1} & a_{n-3} & \cdots & a_1 & \end{array}$$

Each subsequent row is produced from the previous two as

$$s^{n-2} \mid b_{n-2} \quad b_{n-4} \quad \ldots \quad b_0$$

where

$$\begin{aligned} b_{n-2} &= \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} \\ b_{n-4} &= \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}} \end{aligned}$$

and so forth, using zero where no table entry is present. Spectral stability conditions are then determined by requiring all entries in the first column to have the same sign. It should be immediately apparent that this method fails for Hamiltonian systems because the second row will always contain only zeroes. This is indicative of roots occurring in pairs which are symmetric about the origin. An extension to Routh's method allows us to recover. A zero row is replaced by the coefficients of the derivative of the auxiliary polynomial formed from the elements of the previous row. Thus, for a Hamiltonian system with characteristic polynomial (neglecting zero roots)

$$a_n s^n + a_{n-2}s^{n-2} + \cdots + a_2 s^2 + a_0, \tag{3.27}$$

the characteristic polynomial is the auxiliary polynomial for the first row and its derivative is

$$n a_n s^{n-1} + (n-2)a_{n-2}s^{n-3} + \cdots + 2a_2 s \tag{3.28}$$

which supplies the coefficients used for the second row. This procedure is repeated any time a zero row occurs. Again, stability conditions are determined by requiring all entries in the first column to have the same sign. A reaonably complete discussion of Routh's method and the extension for zero rows is presented by D'Azzo and Houpis [21:185–191].

An alternative method based on Sturm sequences has recently been explored by Howard and Mackay [37, 36] and Howard [38, 39] which exploits the even form of the characteristic polynomial. Essentially, we form a *reduced characteristic polynomial* $Q(\sigma)$ by substituting $\sigma = -s^2$ into the characteristic polynomial. For an nth-order Hamiltonian system, this results in a reduced characteristic polynomial of order $n/2$. For stability, we require the eigenvalues of the Hamiltonian system to lie on the imaginary axis. The equivalent requirement is that the roots of the reduced polynomial lie on the positive real axis. To determine the conditions for meeting this requirement, we apply Sturm's method. This method consists of defining a sequence of functions $\{F_k(\sigma)\}$ given by

$$\begin{aligned} F_0(\sigma) &= Q(\sigma) \\ F_1(\sigma) &= Q'(\sigma) \\ F_k(\sigma) &= G_k(\sigma)F_{k-1}(\sigma) - F_{k-2}(\sigma) \end{aligned}$$

where $k = 2, \ldots, r$ and $r \leq n/2$. $G_k(\sigma)$ is the quotient when $F_{k-2}(\sigma)$ is divided by $F_{k-1}(\sigma)$ and $F_k(\sigma)$ is the remainder. The final function in the sequence, $F_r(\sigma)$ is a constant. For spectral stability, we require $r = n/2$ and, for $j = 0, 1, \ldots, n/2 - 1$,

$$\begin{aligned} (-1)^{n/2-j}F_j(0) \geq 0, \\ F_j(+\infty) \geq 0, \\ F_{n/2} \geq 0. \end{aligned}$$

The details of this procedure are presented, along with the calculations for an arbitrary sixth-order Hamiltonian system, by Howard and Mackay [36]. It should be pointed out that Routh's method is also derived from an analysis of Sturm sequences and, in some sense, the two methods are equivalent. The use of Sturm sequences in the development of Routh's method is described in Chapter XV of Gantmacher [28]. It would be of interest

Table 3.1 Spectral Stability Conditions
$n = 2 : P(s) = s^2 + A_0$
$A_0 \geq 0$
$n = 4 : P(s) = s^4 + A_2 s^2 + A_0$
$A_0 \geq 0$ $A_2 \geq 0$ $A_2^2 - 4A_0 \geq 0$
$n = 6 : P(s) = s^6 + A_4 s^4 + A_2 s^2 + A_0$
$A_0 \geq 0$ $A_2 \geq 0$ $A_2 A_4 - 9A_0 \geq 0$ $A_4^2 - 3A_2 \geq 0$ $-27A_0^2 - 4A_2^3 + 18A_0 A_2 A_4 + A_2^2 A_4^2 - 4A_0 A_4^3 \geq 0$
$n = 8 : P(s) = s^8 + A_6 s^6 + A_4 s^4 + A_2 s^2 + A_0$
$A_0 \geq 0$ $A_2 \geq 0$ $3A_6^2 - 8A_4 \geq 0$ $A_6 A_2 - 16A_0 \geq 0$ $A_6^2 A_4 A_2 - 4A_4^2 A_2 + 3A_6 A_2^2 - 9A_6^3 A_0 + 32A_6 A_4 A_0 - 48A_2 A_0 \geq 0$ $A_6^2 A_4^2 - 4A_4^3 - 3A_6^3 A_2 + 14A_6 A_4 A_2 - 18A_2^2 - 6A_6^2 A_0 + 16A_4 A_0 \geq 0$ $A_6^2 A_4^2 A_2^2 - 4A_4^3 A_2^2 - 4A_6^3 A_2^3 + 18A_6 A_4 A_2^3 - 27A_2^4$ $- 4A_6^2 A_4^3 A_0 + 16A_4^4 A_0 + 18A_6^3 A_4 A_2 A_0 - 80A_6 A_4^2 A_2 A_0 - 6A_6^2 A_2^2 A_0$ $+ 144A_4 A_2^2 A_0 - 27A_6^4 A_0^2 + 144A_6^2 A_4 A_0^2 - 128A_4^2 A_0^2 - 192A_6 A_2 A_0^2 + 256A_0^3 \geq 0$

to explore this relationship further and clarify the advantages and disadvantages of one method over the other.

The Hamiltonian systems considered in this dissertation are all ninth-order, and, hence, we will never be concerned with polynomials greater than eighth order in s or fourth order in σ. Table 3.1 presents the spectral stability conditions derived by the method of Howard and Mackay for second-, fourth-, sixth-, and eighth-order polynomials. Recall that each coefficient is a function of the phase variables and physical parameters describing the system. Typically, the stability conditions will be further reduced to expressions involving these variables.

3.3.2 Nonlinear Stability.

Unconstrained Minimum Methods. For finite-dimensional systems, formal stability implies nonlinear stability. Recall that formal stability signifies the existence of a conserved quantity for which the first variation is zero at the relative equilibrium and the second variation is positive- or negative-definite. Such a function is called a Liapunov function. The difficulty in proving nonlinear stability lies in determining a suitable Liapunov function.

Direct Consideration of the Variational Lagrangian. The variational Lagrangian $F(\mathbf{z}) = H(\mathbf{z}) - \sum_{i=1}^{m} \mu_i C_i(\mathbf{z})$ is a candidate Liapunov function for a Hamiltonian system since we have already shown that setting the first variation equal to zero gives precisely the conditions for relative equilibrium. Furthermore, because it is a linear combination of first integrals, it is also conserved. Finally, we have already computed the second variation as part of computing the linear system matrix $\mathbf{A}(\mathbf{z}_e) = \mathbf{J}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)$. Therefore, direct examination to determine positive- or negative-definiteness provides an immediate test for nonlinear stability with little more computational effort. This test provides sufficient conditions for nonlinear stability; however, the conditions are not very sharp and this method proves to be of limited utility.

Energy-Casimir Method. The variational Lagrangian is suggestive of a large class of candidate relative equilibria. If $C(\mathbf{z})$ is a Casimir function or other first integral, then for any smooth scalar function $\phi(x)$ we find that $\phi(C(\mathbf{z}))$ is also a first integral (by the chain rule). Thus, any function of the form

$$H_\phi(\mathbf{z}) = H(\mathbf{z}) + \sum_{i=1}^{m} \phi_i(C_i(\mathbf{z})) \tag{3.29}$$

is a candidate Liapunov function. This may be generalized even further if we consider a scalar function $\Phi(x_0, x_1, \ldots, x_m)$ of $m+1$ variables. Then the class of functions

$$H_\Phi(\mathbf{z}) = \Phi(H(\mathbf{z}), C_1(\mathbf{z}), \ldots, C_m(\mathbf{z})) \tag{3.30}$$

are also candidate Liapunov functions. This latter class includes functions of the type $H_\phi(\mathbf{z})$ but also admits functions which include terms containing products of integrals. The cost of this greater generality is a more difficult analysis. Therefore, although we will start our analysis considering the most general class, H_Φ, it will sometimes be advantageous to consider when the simplification to class H_ϕ will be sufficient to find a suitable Liapunov function.

The first variation specifies conditions on the first derivative of the scalar function(s). These conditions are then substituted into the second variation which is required to be positive- or negative-definite to derive conditions on the second derivatives. If a scalar function (or functions) can be identified which satisfy these conditions, then nonlinear stability has been proven. Extensive details of the application of this method are presented by Holm *et al* [35].

Constrained Minimum Methods. In the preceding analysis of the variational Lagrangian, we may take a variational approach and view the requirement for the second variation to be positive- or negative-definite as equivalently requiring that the relative equilibrium be an unconstrained minimum or maximum of the variational Lagrangian. This is a direct extension of the Lagrange-Dirichlet Theorem for canonical systems which states that a nondegenerate minimum or maximum of the Hamiltonian implies a stable equilibrium. However, as we described earlier, the relative equilibrium may also be viewed as a constrained extremum of the Hamiltonian subject to fixed values of the Casimir functions. The requirement to be a constrained minimum or maximum is less restrictive and thus might be successfully applied to a larger class of relative equilibria. Essentially, rather than considering general perturbations in phase space, this restricts consideration to perturbations on $TM|_{\mathbf{z}_e}$, the tangent space to the invariant manifold $M = \{\mathbf{z} : C_i(\mathbf{z}) = C_i(\mathbf{z}_e), i = 1, \ldots, m\}$ at the relative equilibrium, which may be identified with $\mathcal{R}(\mathbf{A}(\mathbf{z}_e))$, the range of the linearized system matrix. Then, by a theory due to Hestenes,[5] we are assured of the existence

[5]See Lemma 1 of Maddocks and Sachs [63].

of a suitable Liapunov function and may conclude full nonlinear stability when $\mathbf{z}_e$ is a constrained minimum or maximum of the Hamiltonian.[6] Two approaches have evolved from this realization. The first involves direct compution of the projection onto the constraint manifold. The second, the Lagrange Multiplier Approach, has been investigated by Maddocks [62, 61] and Maddocks and Sachs [63] with some success. We briefly summarize these approaches here.

Projection Method. Without loss of generality, we seek a constrained *minimum.* The second-variation condition may be stated as: For all $\mathbf{h} \in TM|_{\mathbf{z}_e}$ such that $\mathbf{h} \neq 0$ (*i.e.*, for all $\mathbf{h} \neq 0$ such that $\nabla C_i(\mathbf{z}_e) \cdot \mathbf{h} = 0$ $(i = 1, \ldots, m)$),

$$\mathbf{h} \cdot \nabla^2 F(\mathbf{z}_e)\mathbf{h} > 0. \tag{3.31}$$

In the absence of symmetries, the range of $\mathbf{A}(\mathbf{z})$ is identically the range of $\mathbf{J}(\mathbf{z})$, and conversely, the nullspace of $\mathbf{A}^\mathsf{T}(\mathbf{z})$ is identically the nullspace of $\mathbf{J}(\mathbf{z})$. Introducing the orthogonal projection matrix, $\mathbf{P}(\mathbf{z})$, onto the range of $\mathbf{A}(\mathbf{z})$, we may write the second-order condition as (recall that $\nabla C_i(\mathbf{z}_e)$ span $\mathcal{N}(\mathbf{J}(\mathbf{z}_e))$, the nullspace of $\mathbf{J}(\mathbf{z}_e)$): For all $\mathbf{h}$ such that $\mathbf{P}(\mathbf{z}_e)\mathbf{h} \neq \mathbf{0}$,

$$\mathbf{h} \cdot \mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)\mathbf{h} > 0. \tag{3.32}$$

Let $\mathbf{Q}(\mathbf{z})$ be the projection onto $\mathcal{N}\left(\mathbf{A}^\mathsf{T}(\mathbf{z})\right) = \mathcal{N}(\mathbf{J}(\mathbf{z}))$ given by

$$\mathbf{Q}(\mathbf{z}) = \mathbf{K}(\mathbf{z})\left(\mathbf{K}^\mathsf{T}(\mathbf{z})\mathbf{K}(\mathbf{z})\right)^{-1}\mathbf{K}^\mathsf{T}(\mathbf{z}) \tag{3.33}$$

[6]We present the case where all the known integrals are Casimir functions. For systems with symmetry integrals, a similar result applies; however, the analysis must be revised to account for the additional constraints and the function for which we seek a constrained extremum is no longer the Hamiltonian. The details of this analysis are presented by example in Chapters 6 and 7 for an axisymmetric rigid body in a central gravitational field.

where $\mathbf{K}(\mathbf{z})$ is the matrix whose columns span the nullspace,

$$\mathbf{K}(\mathbf{z}) = \begin{bmatrix} \nabla C_1(\mathbf{z}) & \dots & \nabla C_m(\mathbf{z}) \end{bmatrix}. \tag{3.34}$$

Then the projection onto the range is given by

$$\mathbf{P}(\mathbf{z}) = \mathbf{1} - \mathbf{Q}(\mathbf{z}). \tag{3.35}$$

We could also have computed $\mathbf{P}(\mathbf{z})$ directly by using row-reduction of either $\mathbf{A}^{\mathsf{T}}(\mathbf{z})$ or $\mathbf{J}(\mathbf{z})$ to generate a basis for $\mathcal{R}(\mathbf{A}(\mathbf{z}))$. We may then compute the *projected Hessian matrix*

$$\mathbf{S}(\mathbf{z}_e) = \mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e). \tag{3.36}$$

This $n \times n$ matrix will have m zero eigenvalues associated with the nullspace. If the remaining eigenvalues are all positive, then $\mathbf{z}_e$ is a constrained minimum and the relative equilibrium is nonlinearly stable.

Lagrange Multiplier Method. Maddocks and Sachs [63] point out that the projection method may be computationally difficult for large n unless there is a simple relationship between the range of $\mathbf{J}(\mathbf{z}_e)$ and the eigenvectors of $\nabla^2 F(\mathbf{z}_e)$. They take an alternative approach to determining whether the relative equilibrium is a constrained minimum which relies on the fact that the relative equilibrium may be embedded in a family of equilibria associated with different values of the Casimir functions. Assuming $\nabla^2 F(\mathbf{z}_e)$ is nonsingular, we may determine a relationship for $\mathbf{z}_e$ as a function of the Lagrange multipliers, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_m)$. Then, the relative equilibrium corresponding to $\mathbf{z}_e$ is embedded in the surface $F(\mathbf{z}_e(\boldsymbol{\mu}))$. Maddocks [61] has shown that the requirement (3.31) is satisfied if and only if the number of positive principal curvatures of the solution surface $F(\mathbf{z}_e(\boldsymbol{\mu}))$ at the relative equilibrium point $\boldsymbol{\mu}_e$ equals the number of negative eigenvalues of $\nabla^2 F(\mathbf{z}_e)$. The interested reader is referred to [63] for a very readable presentation of the details. This method turns out to have limited utility for our applications because the Hessian of F is

singular. In general, we will apply one of the previous three methods to analyze nonlinear stability.

IV. Reduction of the Two-Body Problem

In this chapter, we develop the Hamiltonian system for the motion of two arbitrary rigid bodies moving under the influence of their mutual gravitational attraction. We then specialize to the case when one of the bodies has a spherically symmetric mass distribution. The Hamiltonian system in this case can be reduced to a form which asymptotically approaches that of a free rigid body moving in a central gravitational field as the mass of the second body gets small with respect to the mass of the spherical body. However, unlike the classical two-particle problem, we do not find an exact equivalence. When the second body is also assumed to have a spherically symmetric mass distribution, a *pseudo-equivalence* can be found which is valid in most regions of phase space.

The reduction of the two-body problem to a noncanonical system is a new development[1] which follows directly from the reduction presented by Wang *et al* [104] for a rigid body in a central gravitational field. To our knowledge, the question of an equivalence between these two problems has not been addressed previously. Thus, our proof that no equivalence exists and our initial investigations into the matter of pseudo-equivalence are new contributions.

4.1 Hamiltonian Formulation of Two-Body Problem

4.1.1 Inertial Coordinates. Consider a system comprised of two rigid bodies, $\mathcal{B}_0$ and $\mathcal{B}_1$, which are acted upon by their mutual gravitational attraction and which experience no external forces or moments. Let $\mathcal{F}_i$ be an inertial reference frame and let $\boldsymbol{\xi}_j = (\xi_j, \eta_j, \zeta_j)$ be the coordinates[2] of the center of mass, C_j, of body $\mathcal{B}_j$ where $j = 0, 1$. Let $\mathcal{F}_{p_j}$ be a centroidal principal frame fixed in body $\mathcal{B}_j$ and let $\boldsymbol{\theta}_j = (\psi_j, \theta_j, \phi_j)$ be the classical Euler angles describing the orientation of body frame $\mathcal{F}_{p_j}$ with respect to the inertial frame where ψ_j is the spin angle, θ_j is the nutation angle, and ϕ_j is the precession angle. Denote the

[1]Independent of the current work, this problem has recently been treated by Maciejewski [59] in a more general form than presented here. He expresses the general system in noncanonical variables whereas here we apply reduction through canonical transformations and assume a spherical primary prior to expressing a noncanonical form.

[2]As a matter of convenience, we use the same notation to represent both an ordered set and the corresponding column vector which we treat as interchangeable.

direction cosine matrix for transformation from principal body frame $\mathcal{F}_{p_j}$ to the inertial frame $\mathcal{F}_i$ by $\mathbf{B}_j$. Then the parameterization of $\mathbf{B}_j$ in terms of $\boldsymbol{\theta}_j$ is

$$\mathbf{B}_j(\boldsymbol{\theta}_j) = \begin{bmatrix} \mathrm{c}\,\psi_j\,\mathrm{c}\,\phi_j - \mathrm{s}\,\psi_j\,\mathrm{c}\,\theta_j\,\mathrm{s}\,\phi_j & -\mathrm{s}\,\psi_j\,\mathrm{c}\,\phi_j - \mathrm{c}\,\psi_j\,\mathrm{c}\,\theta_j\,\mathrm{s}\,\phi_j & \mathrm{s}\,\theta_j\,\mathrm{s}\,\phi_j \\ \mathrm{c}\,\psi_j\,\mathrm{s}\,\phi_j + \mathrm{s}\,\psi_j\,\mathrm{c}\,\theta_j\,\mathrm{c}\,\phi_j & -\mathrm{s}\,\psi_j\,\mathrm{s}\,\phi_j + \mathrm{c}\,\psi_j\,\mathrm{c}\,\theta_j\,\mathrm{c}\,\phi_j & -\mathrm{s}\,\theta_j\,\mathrm{c}\,\phi_j \\ \mathrm{s}\,\psi_j\,\mathrm{s}\,\theta_j & \mathrm{c}\,\psi_j\,\mathrm{s}\,\theta_j & \mathrm{c}\,\theta_j \end{bmatrix} \tag{4.1}$$

where we have used c and s as abbreviations for cos and sin. Let $\boldsymbol{\omega}_j$ be the angular velocity of body $\mathcal{B}_j$ expressed in inertial frame $\mathcal{F}_i$ and let $\boldsymbol{\Omega}_j = \mathbf{B}_j^{\mathsf{T}}\boldsymbol{\omega}_j$ be the angular velocity expressed in body frame $\mathcal{F}_{p_j}$. The body frame angular velocity $\boldsymbol{\Omega}_j$ is related to the Euler angles by[3]

$$\boldsymbol{\Omega}_j = \mathbf{S}_j\dot{\boldsymbol{\theta}}_j \tag{4.2}$$

where

$$\mathbf{S}_j(\boldsymbol{\theta}_j) = \begin{bmatrix} 0 & \cos\psi_j & \sin\psi_j\sin\theta_j \\ 0 & -\sin\psi_j & \cos\psi_j\sin\theta_j \\ 1 & 0 & \cos\theta_j \end{bmatrix}. \tag{4.3}$$

Note that $\mathbf{S}_j$ is nonsingular for $\sin\theta_j \neq 0$ so that we may invert it to express the time derivatives of the Euler angles as functions of $\boldsymbol{\Omega}_j$. The inverse is

$$\mathbf{S}_j^{-1}(\boldsymbol{\theta}_j) = \begin{bmatrix} -\sin\psi_j\cos\theta_j/\sin\theta_j & -\cos\psi_j\cos\theta_j/\sin\theta_j & 1 \\ \cos\psi_j & -\sin\psi_j & 0 \\ \sin\psi_j/\sin\theta_j & \cos\psi_j/\sin\theta_j & 0 \end{bmatrix}. \tag{4.4}$$

Also note that $\mathbf{S}_j$ and $\mathbf{S}_j^{-1}$ are independent of the precession angle ϕ_j.

[3] See, *e.g.*, Hughes [40:26–27].

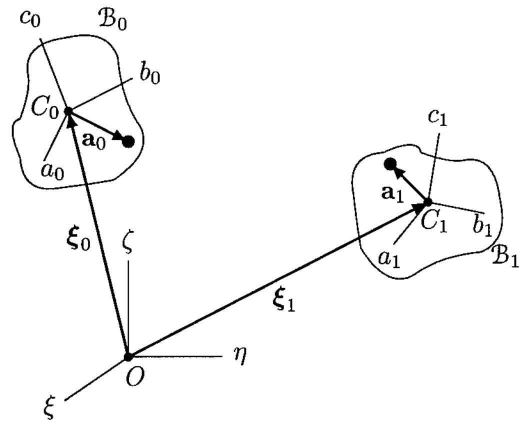


Figure 4.1 Two-Body Configuration

The kinetic energy of the system is

$$\begin{aligned} T &= \sum_{j=0}^{1} \left(\frac{1}{2} m_j \dot{\boldsymbol{\xi}}_j^{\mathsf{T}} \dot{\boldsymbol{\xi}}_j + \frac{1}{2} \boldsymbol{\Omega}_j^{\mathsf{T}} \mathbf{I}_j \boldsymbol{\Omega}_j \right) \\ &= \sum_{j=0}^{1} \left(\frac{1}{2} m_j \dot{\boldsymbol{\xi}}_j^{\mathsf{T}} \dot{\boldsymbol{\xi}}_j + \frac{1}{2} \dot{\boldsymbol{\theta}}_j^{\mathsf{T}} \mathbf{S}_j^{\mathsf{T}} \mathbf{I}_j \mathbf{S}_j \dot{\boldsymbol{\theta}}_j \right) \end{aligned} \tag{4.5}$$

where m_j and $\mathbf{I}_j$ are the mass and the inertia tensor corresponding to body $\mathcal{B}_j$. The inertia tensor is expressed in the body frame. The potential energy of the system is

$$V = -G \int_{\mathcal{B}_1} dm_1 \int_{\mathcal{B}_0} dm_0 \frac{1}{|(\boldsymbol{\xi}_1 + \mathbf{B}_1 \mathbf{a}_1) - (\boldsymbol{\xi}_0 + \mathbf{B}_0 \mathbf{a}_0)|} \tag{4.6}$$

where G is the universal gravitational constant and $\mathbf{a}_j$ is the position of differential mass element dm_j in the appropriate body frame as shown in Figure 4.1. We consider only the case where the bodies do not contact each other. This is assured if we require

$$|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_0| > \max\{|\mathbf{a}_0|\} + \max\{|\mathbf{a}_1|\} \qquad \mathbf{a}_i \in \mathcal{B}_i. \tag{4.7}$$

Any conclusions must be limited to regions of phase space satisfying this restriction in order to be physically meaningful.

Each body has six degrees of freedom and we choose

$$\mathbf{q} = (q_1, q_2, \ldots, q_{12}) = (\boldsymbol{\xi}_0, \boldsymbol{\theta}_0, \boldsymbol{\xi}_1, \boldsymbol{\theta}_1) \tag{4.8}$$

as our generalized coordinates. The kinetic energy is a homogeneous quadratic in the generalized velocities and the potential is a function of only the coordinates. Thus, the system is natural and the Hamiltonian is the total energy

$$H = T + V \tag{4.9}$$

expressed in terms of the coordinates and the conjugate momenta

$$\mathbf{p} = (p_1, p_2, \ldots, p_{12}) = (\mathbf{p}_{\xi_0}, \mathbf{p}_{\theta_0}, \mathbf{p}_{\xi_1}, \mathbf{p}_{\theta_1}). \tag{4.10}$$

The momenta are given by

$$\mathbf{p}_{\xi_j} = \left(\frac{\partial T}{\partial \dot{\boldsymbol{\xi}}_j}\right)^{\mathsf{T}} = m_j \dot{\boldsymbol{\xi}}_j \tag{4.11a}$$

$$\mathbf{p}_{\theta_j} = \left(\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}_j}\right)^{\mathsf{T}} = \mathbf{S}_j^{\mathsf{T}} \mathbf{I}_j \mathbf{S}_j \dot{\boldsymbol{\theta}}_j. \tag{4.11b}$$

The momenta conjugate to $\boldsymbol{\xi}_j$ are the inertial components of the linear momentum of $\mathcal{B}_j$ while the momenta conjugate to $\boldsymbol{\theta}_j$ are the projections of the angular momentum of $\mathcal{B}_j$ onto the nonorthogonal axes about which the Euler angles are defined. The Hamiltonian may then be expressed as

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \mathbf{p} \cdot \mathbf{D} \mathbf{p} + V(\mathbf{q}) \tag{4.12}$$

where

$$\mathbf{D}(\mathbf{q}) = \begin{bmatrix} \frac{1}{m_0}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_0^{-1}\mathbf{I}_0^{-1}\mathbf{S}_0^{-\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{m_1}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}_1^{-1}\mathbf{I}_1^{-1}\mathbf{S}_1^{-\mathsf{T}} \end{bmatrix}. \tag{4.13}$$

The superscript $-\mathsf{T}$ indicates the inverse of the transpose and $\mathbf{0}$ and $\mathbf{1}$ denote the zero and identity matrices of appropriate dimension. Let $\mathbf{z} = (\mathbf{q}, \mathbf{p})$. Then the equations of motion are given as

$$\dot{\mathbf{z}} = \mathbf{J}\nabla_z H(\mathbf{z}) \tag{4.14}$$

where $\mathbf{J}$ is the standard symplectic matrix

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}. \tag{4.15}$$

This is a twenty-fourth-order system of differential equations describing the translational and rotational motion of the two-body system.

4.1.2 Jacobi Coordinates. We next transform the system to a new set of coordinates. We use a canonical transformation which is an identity transformation in the rotational coordinates, but which transforms the translational coordinates to Jacobi coordinates.[4] The Jacobi coordinates for a system of two bodies consist of the inertial coordinates of the center of mass of the system and the coordinates of one mass relative to the other:

$$\boldsymbol{\lambda}_0 = \frac{m_0\boldsymbol{\xi}_0 + m_1\boldsymbol{\xi}_1}{m_0 + m_1} \tag{4.16a}$$

$$\boldsymbol{\lambda}_1 = \boldsymbol{\xi}_1 - \boldsymbol{\xi}_0. \tag{4.16b}$$

[4]See, *e.g.*, Meyer and Hall [71:92–94] or Battin [10:398–400].

Let $\bar{\mathbf{z}} = (\bar{\mathbf{q}}, \bar{\mathbf{p}}) = ((\boldsymbol{\lambda}_0, \boldsymbol{\theta}_0, \boldsymbol{\lambda}_1, \boldsymbol{\theta}_1), (\mathbf{p}_{\lambda_0}, \mathbf{p}_{\theta_0}, \mathbf{p}_{\lambda_1}, \mathbf{p}_{\theta_1}))$ be the new phase variables and let $m = m_0 + m_1$ be the total system mass. The generating function for this canonical transformation is

$$F_2(\mathbf{q}, \bar{\mathbf{p}}) = \mathbf{p}_{\lambda_0} \cdot \left(\frac{m_0}{m}\boldsymbol{\xi}_0 + \frac{m_1}{m}\boldsymbol{\xi}_1\right) + \mathbf{p}_{\theta_0} \cdot \boldsymbol{\theta}_0 + \mathbf{p}_{\lambda_1} \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_0) + \mathbf{p}_{\theta_1} \cdot \boldsymbol{\theta}_1 \tag{4.17}$$

which gives

$$\mathbf{p}_{\xi_0} = \frac{\partial F_2}{\partial \boldsymbol{\xi}_0} = \frac{m_0}{m}\mathbf{p}_{\lambda_0} - \mathbf{p}_{\lambda_1} \tag{4.18a}$$

$$\mathbf{p}_{\xi_1} = \frac{\partial F_2}{\partial \boldsymbol{\xi}_1} = \frac{m_1}{m}\mathbf{p}_{\lambda_0} + \mathbf{p}_{\lambda_1} \tag{4.18b}$$

for the translational momenta along with an identity transformation in the rotational momenta. Solving for the new momenta, $\mathbf{p}_{\lambda_0}$ and $\mathbf{p}_{\lambda_1}$, we find

$$\mathbf{p}_{\lambda_0} = \mathbf{p}_{\xi_0} + \mathbf{p}_{\xi_1} = m\dot{\boldsymbol{\lambda}}_0 \tag{4.19a}$$

$$\mathbf{p}_{\lambda_1} = -\frac{m_1}{m}\mathbf{p}_{\xi_0} + \frac{m_0}{m}\mathbf{p}_{\xi_1} = \mu\dot{\boldsymbol{\lambda}}_1 \tag{4.19b}$$

where $\mu = m_0 m_1/m$ is the reduced mass. We may also solve for the old translational coordinates in terms of the new ones to get

$$\boldsymbol{\xi}_0 = \boldsymbol{\lambda}_0 - \frac{\mu}{m_0}\boldsymbol{\lambda}_1 \tag{4.20a}$$

$$\boldsymbol{\xi}_1 = \boldsymbol{\lambda}_0 + \frac{\mu}{m_1}\boldsymbol{\lambda}_1. \tag{4.20b}$$

Since F_2 is independent of time, the new Hamiltonian is

$$\begin{aligned} \bar{H}(\bar{\mathbf{q}}, \bar{\mathbf{p}}) &= H(\mathbf{q}(\bar{\mathbf{q}}), \mathbf{p}(\bar{\mathbf{p}})) \\ &= T(\mathbf{q}(\bar{\mathbf{q}}), \mathbf{p}(\bar{\mathbf{p}})) + V(\mathbf{q}(\bar{\mathbf{q}})) \\ &= \frac{1}{2}\bar{\mathbf{p}} \cdot \bar{\mathbf{D}}\bar{\mathbf{p}} + \bar{V}(\bar{\mathbf{q}}) \end{aligned} \tag{4.21}$$

where

$$\bar{\mathbf{D}}(\bar{\mathbf{q}}) = \begin{bmatrix} \frac{1}{m}\mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_0^{-1}\mathbf{I}_0^{-1}\mathbf{S}_0^{-\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \frac{1}{\mu}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{S}_1^{-1}\mathbf{I}_1^{-1}\mathbf{S}_1^{-\mathsf{T}} \end{bmatrix} \tag{4.22}$$

and

$$\bar{V}(\bar{\mathbf{q}}) = -G \int_{\mathcal{B}_1} dm_1 \int_{\mathcal{B}_0} dm_0 \frac{1}{|\boldsymbol{\lambda}_1 + \mathbf{B}_1\mathbf{a}_1 - \mathbf{B}_0\mathbf{a}_0|}. \tag{4.23}$$

The potential is independent of $\boldsymbol{\lambda}_0$. The new equations of motion are given by

$$\dot{\bar{\mathbf{z}}} = \mathbf{J}\nabla_{\bar{z}}\bar{H} \tag{4.24}$$

which is still a twenty-fourth-order system. In the new phase variables, the restriction given in Equation (4.7) becomes

$$|\boldsymbol{\lambda}_1| > \max\{|\mathbf{a}_0|\} + \max\{|\mathbf{a}_1|\} \qquad \mathbf{a}_i \in \mathcal{B}_i \tag{4.25}$$

which identifies the regions of phase space that are physically possible.

4.1.3 Integrals of Motion and Partial Reduction. Duboshin [24, 25] has shown that for the general problem of n finite bodies there are ten first integrals, analogous to the ten first integrals of the classical n-body problem in which bodies are treated as point masses. There are six associated with conservation of linear momentum, three associated with conservation of angular momentum about the system center of mass, and one associated with conservation of energy.

The energy integral is immediately apparent since the Hamiltonian is independent of time. The integrals related to conservation of linear momentum are also readily apparent when the system is expressed in Jacobi coordinates. Note that $\boldsymbol{\lambda}_0$ is ignorable, implying

that the conjugate momentum $\mathbf{p}_{\lambda_0} = m\dot{\boldsymbol{\lambda}}_0$ is a first integral. A second integration yields

$$\boldsymbol{\lambda}_0(t) = \frac{t}{m}\mathbf{p}_{\lambda_0} + \mathbf{c} \tag{4.26}$$

where $\mathbf{c}$ is a vector of constants of integration representing the coordinates of the center of mass at epoch. The remaining three integrals are the components of the system angular momentum in the inertial frame. We omit the proof which is given by Duboshin [24, 25]. In terms of $\bar{\mathbf{z}}$ the angular momentum integral vector is

$$\mathbf{L}(\bar{\mathbf{z}}) = \boldsymbol{\lambda}_0{}^{\times}\mathbf{p}_{\lambda_0} + \mathbf{B}_0\mathbf{S}_0^{-\mathsf{T}}\mathbf{p}_{\theta_0} + \boldsymbol{\lambda}_1{}^{\times}\mathbf{p}_{\lambda_1} + \mathbf{B}_1\mathbf{S}_1^{-\mathsf{T}}\mathbf{p}_{\theta_1} \tag{4.27}$$

where the superscript $\times$ denotes the skew-symmetric matrix form associated with the cross-product in $\mathbb{R}^3$.

Since the center of mass moves with constant velocity relative to an inertial frame, we may choose an inertial frame with the system center of mass as its origin, effectively removing three degrees of freedom. Then we have $\boldsymbol{\lambda}_0(t) \equiv \mathbf{0}$ and $\mathbf{p}_{\lambda_0}(t) \equiv \mathbf{0}$. Let the phase variables for the reduced system be $\tilde{\mathbf{z}} = (\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) = ((\boldsymbol{\lambda}_1, \boldsymbol{\theta}_1, \boldsymbol{\theta}_0), (\mathbf{p}_{\lambda_1}, \mathbf{p}_{\theta_1}, \mathbf{p}_{\theta_0}))$. The new Hamiltonian is

$$\tilde{H}(\tilde{\mathbf{q}}, \tilde{\mathbf{p}}) = \frac{1}{2}\tilde{\mathbf{p}} \cdot \tilde{\mathbf{D}}\tilde{\mathbf{p}} + \tilde{V}(\tilde{\mathbf{q}}) \tag{4.28}$$

where

$$\tilde{\mathbf{D}}(\tilde{\mathbf{q}}) = \begin{bmatrix} \frac{1}{\mu}\mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1^{-1}\mathbf{I}_1^{-1}\mathbf{S}_1^{-\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S}_0^{-1}\mathbf{I}_0^{-1}\mathbf{S}_0^{-\mathsf{T}} \end{bmatrix} \tag{4.29}$$

and $\tilde{V}(\tilde{\mathbf{q}}) = \overline{V}(\bar{\mathbf{q}})$. The equations of motion are

$$\dot{\tilde{\mathbf{z}}} = \mathbf{J}\nabla_{\tilde{z}}\tilde{H}(\tilde{\mathbf{z}}) \tag{4.30}$$

which is an eighteenth-order system. For this reduced system, only four integrals of motion remain: the energy and the three components of total angular momentum given in Equation (4.27) (noting that the first term on the right-hand side vanishes).

4.1.4 One Spherically Symmetric Body. Now suppose that body $\mathcal{B}_0$ has a mass distribution which is spherically symmetric. Newton [78] showed that at all points exterior to such a body the potential function is equal to that of a particle of like mass located at the center of mass of the body. The potential function is then independent of $\boldsymbol{\theta}_0$ which implies there is no net torque acting on body $\mathcal{B}_0$ about its center of mass. Since the moments of inertia are equal (to I_0, say), the body angular momentum is $\boldsymbol{\Pi}_0 = I_0\boldsymbol{\Omega}_0$ and Euler's equations of motion show that the angular velocity and the angular momentum are constant in both the inertial frame $\mathcal{F}_i$ and the body frame $\mathcal{F}_{p0}$. The attitude motion of $\mathcal{B}_0$ is therefore decoupled and we may subtract its constant contribution to the energy and consider just the six degrees of freedom associated with $\boldsymbol{\lambda}_1$ and $\boldsymbol{\theta}_1$. The new Hamiltonian is

$$\widehat{H}(\widehat{\mathbf{q}},\widehat{\mathbf{p}}) = \frac{1}{2}\widehat{\mathbf{p}}\cdot\widehat{\mathbf{D}}\widehat{\mathbf{p}} + \widehat{V}(\widehat{\mathbf{q}}) \tag{4.31}$$

where the coordinate and momentum vectors are $\widehat{\mathbf{q}} = (\boldsymbol{\lambda}_1,\boldsymbol{\theta}_1)$ and $\widehat{\mathbf{p}} = (\mathbf{p}_{\lambda_1},\mathbf{p}_{\theta_1})$ and where

$$\widehat{\mathbf{D}} = \begin{bmatrix} \frac{1}{\mu}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_1^{-1}\mathbf{I}_1^{-1}\mathbf{S}_1^{-\mathsf{T}} \end{bmatrix} \tag{4.32}$$

and

$$\widehat{V}(\widehat{\mathbf{q}}) = -Gm_0\int_{\mathcal{B}_1} dm_1\frac{1}{|\boldsymbol{\lambda}_1+\mathbf{B}_1\mathbf{a}_1|}. \tag{4.33}$$

The equations of motion for this reduced system are then

$$\dot{\widehat{\mathbf{z}}} = \mathbf{J}\nabla_{\widehat{z}}\widehat{H}(\widehat{\mathbf{z}}) \tag{4.34}$$

which is a twelfth-order system. We may also subtract the constant contribution of the rotation of $\mathcal{B}_0$ to the system angular momentum to get the new integral vector

$$\widehat{\mathbf{L}}(\widehat{\mathbf{z}}) = \boldsymbol{\lambda}_1{}^{\times}\mathbf{p}_{\lambda_1} + \mathbf{B}_1\mathbf{S}_1^{-\mathsf{T}}\mathbf{p}_{\theta_1}. \tag{4.35}$$

We have effectively removed the three degrees of freedom associated with the rotation of $\mathcal{B}_0$.

4.1.5 Noncanonical Coordinates. The above canonical system has been reduced to a twelfth-order system from the twenty-fourth-order system we developed at the start of this chapter. We know four independent first integrals for this reduced system: the energy integral and three angular momentum integrals. The last reduction was a result of the spherical symmetry assumption imposed on body $\mathcal{B}_0$. This symmetry caused the attitude dynamics of the spherical body to decouple from the remaining dynamics which form our reduced system. However, the symmetry of $\mathcal{B}_0$ also introduces symmetry into the solutions of the reduced system. If we consider a transformation which takes the phase variables to the body frame for $\mathcal{B}_1$, the result is a reduction to a ninth-order noncanonical system with two known first integrals. Before presenting this transformation, however, we briefly review notation and terminology.

Much of the development which follows builds on the work of Wang *et al* [105] which treats a rigid body in a central gravitational field. However, for clarity we have modified their notation and adopted the practice of Marsden and Ratiu [67] where a small greek letter denotes the inertial frame representation of a quantity while the capital letter denotes the body (or nodal) frame representation. Caution must be exercised in comparing the results given here to those in [105]. Table 4.1 shows the notation for the particular quantities dealt with here. The variables are subscripted as appropriate to indicate association with a given rigid body.

Our terminology may also cause some confusion. It must be emphasized that what we refer to here as the position is relative to $\mathcal{B}_0$ and what we refer to as linear momentum is

Table 4.1 Notation Convention for Phase Variables

Quantity	Inertial	Body	Relation to Canonical Coordinates
Position	$\boldsymbol{\lambda}$	$\boldsymbol{\Lambda}$	$\boldsymbol{\lambda} = \boldsymbol{\lambda}$
Linear Momentum	$\boldsymbol{\sigma}$	$\boldsymbol{\Sigma}$	$\boldsymbol{\sigma} = \mathbf{p}_\lambda$
Angular Momentum	$\boldsymbol{\pi}$	$\boldsymbol{\Pi}$	$\boldsymbol{\Pi} = \mathbf{S}^{-\mathsf{T}}\mathbf{p}_\theta$
Angular Velocity	$\boldsymbol{\omega}$	$\boldsymbol{\Omega}$	$\boldsymbol{\Omega} = \mathbf{I}^{-1}\boldsymbol{\Pi}$

not truly the linear momentum. The true linear momentum for $\mathcal{B}_1$ was given earlier as $\mathbf{p}_{\xi_1}$, the momentum conjugate to the inertial position $\boldsymbol{\xi}_1$. However, we will find it convenient to overlook this difference since, when we approximate the system as a rigid body in a central gravitational field, these terms will regain their usual meaning.

Consider the Hamiltonian equations for the canonical system, which may be expressed as

$$\dot{\boldsymbol{\lambda}}_1 = \frac{1}{\mu}\mathbf{p}_{\lambda_1} \tag{4.36a}$$

$$\dot{\boldsymbol{\theta}}_1 = \mathbf{S}_1^{-1}\mathbf{I}_1^{-1}\mathbf{S}_1^{-\mathsf{T}}\mathbf{p}_{\theta_1} \tag{4.36b}$$

$$\dot{\mathbf{p}}_{\lambda_1} = -\nabla_{\lambda_1}\widehat{V}(\boldsymbol{\lambda}_1, \boldsymbol{\theta}_1) \tag{4.36c}$$

$$\dot{\mathbf{p}}_{\theta_1} = -\nabla_{\theta_1}\widehat{V}(\boldsymbol{\lambda}_1, \boldsymbol{\theta}_1). \tag{4.36d}$$

We may rewrite these equations in terms of the variables $\boldsymbol{\Lambda}_1$, $\boldsymbol{\Sigma}_1$, and $\boldsymbol{\Pi}_1$, along with the direction cosine matrix $\mathbf{B}_1$. Equation (4.36b) may be written more generally in terms of the direction cosine matrix as

$$\dot{\mathbf{B}}_1 = \mathbf{B}_1\boldsymbol{\Omega}_1^{\times} \tag{4.37}$$

which is a form of Poisson's equations for the rotational kinematics of a rigid body. The differential equations for $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Sigma}_1$ may be derived by differentiating the relations $\boldsymbol{\Lambda}_1 = \mathbf{B}_1^{\mathsf{T}}\boldsymbol{\lambda}_1$ and $\boldsymbol{\Sigma}_1 = \mathbf{B}_1^{\mathsf{T}}\boldsymbol{\sigma}_1$ and subsequently incorporating Equations (4.36a), (4.36c), and

(4.37) to give

$$\dot{\mathbf{\Lambda}}_1 = \mathbf{\Lambda}_1{}^{\times}\mathbf{I}_1^{-1}\mathbf{\Pi}_1 + \frac{1}{\mu}\mathbf{\Sigma}_1 \tag{4.38a}$$

$$\dot{\mathbf{\Sigma}}_1 = \mathbf{\Sigma}_1{}^{\times}\mathbf{I}_1^{-1}\mathbf{\Pi}_1 - \nabla_{\Lambda_1}V(\mathbf{\Lambda}_1) \tag{4.38b}$$

where the potential

$$V(\mathbf{\Lambda}_1) = -Gm_0\int_{\mathcal{B}_1} dm_1 \frac{1}{|\mathbf{\Lambda}_1 + \mathbf{a}_1|}. \tag{4.39}$$

is derived from Equation (4.33). Note that by transforming to body frame variables, the potential is now independent of $\boldsymbol{\theta}_1$ (or $\mathbf{B}_1$). A similar derivation for $\mathbf{\Pi}_1$ requires considerably more effort. However, the desired equation is Euler's equation which may be written as

$$\dot{\mathbf{\Pi}}_1 = \mathbf{\Pi}_1{}^{\times}\mathbf{I}_1^{-1}\mathbf{\Pi}_1 + \mathbf{\Lambda}_1{}^{\times}\nabla_{\Lambda_1}V(\mathbf{\Lambda}_1) \tag{4.40}$$

where the latter term is an expression for the gravitational torque (derived in Appendix B).

The differential equations for $\mathbf{\Lambda}_1$, $\mathbf{\Sigma}_1$, and $\mathbf{\Pi}_1$ turn out to be independent of $\mathbf{B}_1$. Thus we may treat this decoupled ninth-order subsystem. The equations for this reduced system may be written in noncanonical Hamiltonian form for the phase variable vector $\mathbf{z} = (\mathbf{\Sigma}_1, \mathbf{\Lambda}_1, \mathbf{\Pi}_1)$ as

$$\dot{\mathbf{z}} = \mathbf{J}(\mathbf{z})\nabla H(\mathbf{z}) \tag{4.41}$$

where the Poisson structure matrix is

$$\mathbf{J}(\mathbf{z}) = \begin{bmatrix} \mathbf{0} & -\mathbf{1} & \mathbf{\Sigma}_1{}^{\times} \\ \mathbf{1} & \mathbf{0} & \mathbf{\Lambda}_1{}^{\times} \\ \mathbf{\Sigma}_1{}^{\times} & \mathbf{\Lambda}_1{}^{\times} & \mathbf{\Pi}_1{}^{\times} \end{bmatrix} \tag{4.42}$$

and the Hamiltonian is

$$H(\mathbf{z}) = \frac{1}{2}\mathbf{\Pi}_1 \cdot \mathbf{I}_1^{-1}\mathbf{\Pi}_1 + \frac{1}{2\mu}\mathbf{\Sigma}_1 \cdot \mathbf{\Sigma}_1 + V(\mathbf{\Lambda}_1) \tag{4.43}$$

which is derived from Equation (4.31).

The Hamiltonian for this system is still the energy integral. In addition, the Poisson structure matrix has a one-dimensional null space from which we derive the Casimir function

$$C(\mathbf{z}) = \frac{1}{2}\left|\mathbf{\Pi}_1 + \mathbf{\Lambda}_1{}^{\times}\mathbf{\Sigma}_1\right|^2 \tag{4.44}$$

which is the magnitude of the system angular momentum vector given in Equation (4.35). We thus have a ninth-order system with two independent first integrals, as claimed. This is the form of the two-body problem which we seek to approximate in the work which follows.

4.2 Comparison to a Free Rigid Body in a Central Gravitational Field

For the classical problem of two particles, there exists a one-to-one correspondence with the problem of a particle moving in a central gravitational field. That is, given a system of two particles of known mass, m_0 and m_1, we can express the equations of motion in Jacobi coordinates and eliminate the three degrees of freedom associated with the motion of the center of mass. The remaining equations of relative motion are precisely the equations of motion for a particle with reduced mass $\mu = m_0 m_1/(m_0 + m_1)$ moving in a central gravitational field with strength $G(m_0 + m_1)$.

It seems reasonable to question whether a similar equivalence exists between the problem of two finite bodies and the problem of a rigid body in a central gravitational field. From the previous sections, it should be clear that it is necessary to assume one of the bodies in the two-body problem is spherical to introduce the symmetry present in the central gravitational field. The resulting Hamiltonian system given in Equation (4.41) has

a form which is identical to that of a rigid body in a central gravitational field as first derived by Wang *et al* [104]. In our notation, this system is given as

$$\dot{\mathbf{z}}_* = \mathbf{J}_*(\mathbf{z}_*)\nabla H_*(\mathbf{z}_*) \tag{4.45}$$

where the phase variable vector is $\mathbf{z}_* = (\boldsymbol{\Sigma}_*, \boldsymbol{\Lambda}_*, \boldsymbol{\Pi}_*)$, the structure matrix is

$$\mathbf{J}_*(\mathbf{z}_*) = \begin{bmatrix} \mathbf{0} & -\mathbf{1} & \boldsymbol{\Sigma}_*{}^{\times} \\ \mathbf{1} & \mathbf{0} & \boldsymbol{\Lambda}_*{}^{\times} \\ \boldsymbol{\Sigma}_*{}^{\times} & \boldsymbol{\Lambda}_*{}^{\times} & \boldsymbol{\Pi}_*{}^{\times} \end{bmatrix}, \tag{4.46}$$

and the Hamiltonian is

$$H_*(\mathbf{z}_*) = \frac{1}{2}\boldsymbol{\Pi}_* \cdot \mathbf{I}_*^{-1}\boldsymbol{\Pi}_* + \frac{1}{2m_*}\boldsymbol{\Sigma}_* \cdot \boldsymbol{\Sigma}_* + V_*(\boldsymbol{\Lambda}_*). \tag{4.47}$$

The potential for this problem is

$$V_*(\boldsymbol{\Lambda}_*) = -G_* \int_{\mathcal{B}_*} dm_* \frac{1}{|\boldsymbol{\Lambda}_* + a_*|} \tag{4.48}$$

where G_* is the gravitational field strength. This Hamiltonian system is obviously of the same form as the Hamiltonian system for the two-body problem. The search for an equivalence becomes a question of existence. That is, given the physical description of the bodies in the two-body problem, is there a finite rigid body and choice of field strength for which the equations of motion in a central gravitational field are identical when we set $\boldsymbol{\Lambda}_* = \boldsymbol{\Lambda}_1 = \boldsymbol{\Lambda}$, $\boldsymbol{\Sigma}_* = \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}_* = \boldsymbol{\Pi}_1 = \boldsymbol{\Pi}$? For such an equivalence to exist, the requirements may be specified in terms of the mass, the inertia tensor, and the potential:

$$m_* = \mu \tag{4.49a}$$

$$\mathbf{I}_* = \mathbf{I}_1 \tag{4.49b}$$

$$V_*(\boldsymbol{\Lambda}) = V(\boldsymbol{\Lambda}) \;\forall\; \boldsymbol{\Lambda} \in \mathbb{R}^3. \tag{4.49c}$$

Let $\mathcal{B}_1$ and the proposed equivalent body $\mathcal{B}_*$ be described by the density distributions $\rho_1(\mathbf{a})$ and $\rho_*(\mathbf{a})$ defined over a sphere $\mathcal{S}$ of radius R centered on the center of mass. Choose R sufficiently large so that the sphere encloses both bodies. Then we require

$$\int_{\mathcal{S}} \rho_*(\mathbf{a}) d\mathbf{a} = \frac{m_0}{m_0 + m_1} \int_{\mathcal{S}} \rho_1(\mathbf{a}) d\mathbf{a} \tag{4.50a}$$

$$\int_{\mathcal{S}} \mathbf{a}^\times \mathbf{a}^\times \rho_*(\mathbf{a}) d\mathbf{a} = \int_{\mathcal{S}} \mathbf{a}^\times \mathbf{a}^\times \rho_1(\mathbf{a}) d\mathbf{a} \tag{4.50b}$$

$$G_* \int_{\mathcal{S}} \frac{1}{|\boldsymbol{\Lambda} + \mathbf{a}|} \rho_*(\mathbf{a}) d\mathbf{a} = G m_0 \int_{\mathcal{S}} \frac{1}{|\boldsymbol{\Lambda} + \mathbf{a}|} \rho_1(\mathbf{a}) d\mathbf{a} \tag{4.50c}$$

or

$$\int_{\mathcal{S}} \left[\rho_*(\mathbf{a}) - \frac{m_0}{m_0 + m_1} \rho_1(\mathbf{a}) \right] d\mathbf{a} = 0 \tag{4.51a}$$

$$\int_{\mathcal{S}} \mathbf{a}^\times \mathbf{a}^\times \left[\rho_*(\mathbf{a}) - \rho_1(\mathbf{a}) \right] d\mathbf{a} = 0 \tag{4.51b}$$

$$\int_{\mathcal{S}} \left[G_* \rho_*(\mathbf{a}) - G m_0 \rho_1(\mathbf{a}) \right] \frac{1}{|\boldsymbol{\Lambda} + \mathbf{a}|} d\mathbf{a} = 0 \tag{4.51c}$$

in order for an equivalence to exist.

We begin by showing that these requirements cannot be satisfied for all $\boldsymbol{\Lambda}$. It is well known that the potential for a rigid body satisfies Laplace's equation

$$\Delta V = \nabla \cdot \nabla V = 0 \tag{4.52}$$

at all points outside the rigid body. This result was given in more general form by Poisson[5] as

$$\Delta V = -4\pi\rho \tag{4.53}$$

which is valid at all points in space. This equation states that there is a one-to-one correspondence between the mass distribution and the potential function and implies that

[5] See, *e.g.*, MacMillan [60:§69].

Equation (4.49c) can only be satisfied if

$$\rho_*(\mathbf{a}) = \frac{GM_0}{G_*}\rho_1(\mathbf{a}). \tag{4.54}$$

That is, the proposed equivalent body is the actual body with the density scaled by $\frac{Gm_0}{G_*}$. Substituting this into Equation (4.50a), we find

$$G_* = G(m_0 + m_1) \tag{4.55}$$

while substitution into Equation (4.50b) leads to

$$G_* = Gm_0. \tag{4.56}$$

Obviously, these results are incompatible and therefore no equivalent body exists.

However, the above analysis is overly restrictive since we have previously limited consideration to only those regions of phase space where the bodies are not in contact. A looser definition of equivalence was assumed when we equated the spherical primary with a mass point, since their potentials are only identical at points outside the spherical body. Similarly, here we are only interested in equivalence for values of $\mathbf{\Lambda}$ sufficiently large so that the body does not contact the primary. If the primary has radius R_0 then the restriction in Equation (4.25) becomes

$$|\mathbf{\Lambda}| > R_0 + \max\{|\mathbf{a}_1|\} \qquad \mathbf{a}_1 \in \mathcal{B}_1. \tag{4.57}$$

This again restricts us to the physically realizable portions of phase space we are interested in. We refer to two systems which are equivalent within this region as *pseudo-equivalent.*

The possibility of pseudo-equivalence still remains. For instance, the equivalence requirements can easily be satisfied when body $\mathcal{B}_1$ has a spherically symmetric mass distribution. We may assume a spherically symmetric pseudo-equivalent body, in which case

requirement (4.51c) reduces to

$$G_* m_* - G m_0 m_1 = 0. \tag{4.58}$$

In light of requirement (4.50a), this is satisfied when the gravitational field strength is $G_* = G\,(m_0 + m_1)$. The volume integral in the mass requirement (4.50a) reduces to a line integral to give the simpler condition

$$4\pi \int_0^R r^2 \rho_*(r) dr = \mu. \tag{4.59}$$

Likewise, the volume integral in the inertia requirement (4.50b) reduces to a line integral. Furthermore, this matrix equation reduces to the scalar equation

$$\frac{8}{3}\pi \int_0^R r^4 \rho_*(r) dr = I_1 \tag{4.60}$$

where $\mathbf{I}_1 = I_1 \mathbf{1}$. These two requirements can always be satisfied if we assume the pseudo-equivalent body is a homogeneous sphere of radius r_0 and density ρ_0. Conditions (4.59) and (4.60) may be solved for r_0 and ρ_0 to give

$$r_0^2 = \frac{5 I_1}{2\mu} \tag{4.61a}$$

$$\rho_0 = \frac{3\mu^{\frac{5}{2}}}{5\sqrt{10} I_1 \pi}. \tag{4.61b}$$

There is no reason to expect that this pseudo-equivalent body is unique. The existence of a pseudo-equivalent body for an arbitrary rigid body remains an open question.

If we consider the limiting case as the mass of $\mathcal{B}_1$ becomes very small relative to the mass of $\mathcal{B}_0$, we find

$$\lim_{m_1/m_0 \to 0} \mu = \lim_{m_1/m_0 \to 0} \frac{m_0 m_1}{m_0 + m_1} = \lim_{m_1/m_0 \to 0} \frac{m_1}{1 + m_1/m_0} = m_1 \tag{4.62}$$

and the equivalent body $\mathcal{B}_*$ is just the original body $\mathcal{B}_1$. The gravitational field strength is $G_* = Gm_0$. This limiting case justifies the use of the central gravitational field approximation for problems where m_1 is many orders of magnitude smaller than m_0 (as in the case of the artificial satellite problem). This turns out to be the first in a hierarchy of approximations we can use in order to simplify the two-body problem. This hierarchy is addressed in the next chapter.

V. *Approximation of the Two-Body Problem*

In this chapter, we further examine approaches to approximation of the two-body problem, establishing a three-dimensional hierarchy of related Hamiltonian systems. All approximations considered here consist of a rigid body moving in a central gravitational field. They differ in the degree of simplifications introduced. The first dimension of the hierarchy is associated with increasing constraints on the trajectory of the satellite; the second dimension is increasing symmetry of the satellite; and the third dimension is decreasing order of approximation of the potential. Primary emphasis is placed on deriving the Hamiltonian systems for the three problems comprising the first dimension. The simplifications introduced by progression along the other two dimensions are then discussed. Finally, we present the entire hierarchy, highlighting those problems which are investigated in this dissertation.

The presentation given here finds its origin in the work of Beletskii [15] whose research examined many of the systems in the hierarchy using classical methods. The relationships between the various systems in the hierarchy can be inferred from the discussion in [15], but here we formally identify those relationships. Of the three problems for which we derive the noncanonical Hamiltonian system, one of the derivations is due to Wang *et al* [104], one is new but virtually identical to the system treated by Maddocks [62], and the third is an entirely new derivation. The treatment of potential approximations is based on the treatment by Wang *et al* [104].

5.1 Trajectory Constraints

In the previous chapter, we used a process of reduction (along with an assumption of spherical symmetry of one body) to transform the Hamiltonian system under consideration from a twenty-fourth-order system with ten known first integrals to a ninth-order system with two known first integrals. We also showed that this system can be approximated as a rigid body moving in a central gravitational field when the mass of the second body (or

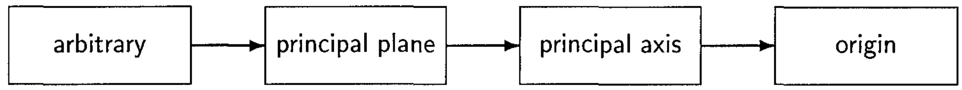


Figure 5.1 Constraints on Location of Center of Mass in Body Frame

satellite) is very small compared to the mass of the spherical body (or primary). We refer to the approximate system as the *free rigid-body system.*

The classical approach to treating the relative equilibria of a satellite moving about a large primary results in a different approximation. In this approach, the series expansions of the force and torque are each truncated after the lead term. The force which results is the same as the force for a particle of equal mass located at the center of mass. The center of mass therefore follows an (orthogonal) Keplerian orbit. If we constrain the center of mass to follow this orbit and consider the resulting attitude motion of the rigid body due to the torques derived from the true potential, we have what we refer to as the *Keplerian system.* Since we are interested in relative equilibria similar to those of the two-body and free rigid-body systems, we only consider circular orbits.

Finally, if we constrain a point in the body to remain fixed in inertial space, we have a system in which the centripetal acceleration effects are removed and only the central gravitational field effects remain. We refer to this as the *fixed-point system.* This system is a generalization of the classical heavy-top problem which treats motion about a fixed point in a uniform gravitational field.

Beletskii [15:11] first identified this categorization for the problems of a rigid body in a central gravitational field. Note that in the free rigid-body and Keplerian systems we assume the point C is the center of mass, while no such assumption is made for the fixed-point system. In fact we could expand this dimension of the hierarchy to two dimensions by treating a series of possible constraints on the position of the center of mass relative to the origin of the body frame, C, as shown in Figure 5.1. Maddocks [62] considered the same series of constraints in a noncanonical treatment of the heavy-top problem. However, by

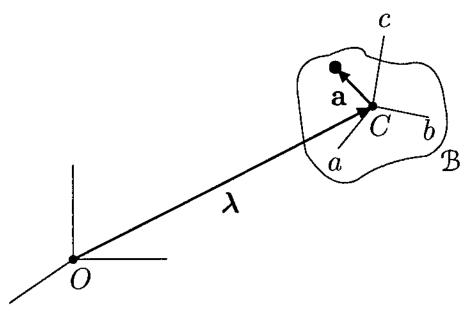


Figure 5.2 Configuration of Free Rigid-Body System

restricting consideration to only those problems where C is either fixed or is the center of mass, the relationship between angular momentum and torque about C takes the simpler form

$$\dot{\boldsymbol{\pi}} = \mathbf{g}_c \tag{5.1}$$

in the inertial frame. For the free rigid-body system, the choice of the point C as the center of mass is simply a matter of convenience. However, for the constrained systems, the choice of C changes the dynamics.

5.1.1 Free Rigid-Body System. For an unconstrained rigid body in a central gravitational field, the configuration is described by the inertial position of the point C at the origin of the body frame and the orientation of the body frame relative to the inertial frame. We assume the frame is centroidal. The system has six degrees of freedom — three associated with translation of the body and three associated with rotation. As before, let $\boldsymbol{\lambda}$ denote the position of the center of mass and $\boldsymbol{\theta}$ denote the Euler angles. This configuration is shown in Figure 5.2.

Following a development similar to that presented in Chapter 4 for the two-body problem, a twelfth-order canonical system can be derived for which the total energy and the angular momentum about the center of attraction provide four first integrals. We instead proceed directly to the noncanonical system from Newton's Second Law and Equation (5.1).

In the inertial frame we have

$$\dot{\sigma} = \mathbf{f} \tag{5.2a}$$

$$\dot{\lambda} = \frac{1}{m}\sigma \tag{5.2b}$$

$$\dot{\pi} = \mathbf{g}_c \tag{5.2c}$$

where $\mathbf{f}$ and $\mathbf{g}_c$ are the inertial force and torque about the center of mass. In the body frame these equations become

$$\dot{\mathbf{\Sigma}} = \mathbf{\Sigma}^{\times}\mathbf{I}^{-1}\mathbf{\Pi} + \mathbf{F} \tag{5.3a}$$

$$\dot{\mathbf{\Lambda}} = \mathbf{\Lambda}^{\times}\mathbf{I}^{-1}\mathbf{\Pi} + \frac{1}{m}\mathbf{\Sigma} \tag{5.3b}$$

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi}^{\times}\mathbf{I}^{-1}\mathbf{\Pi} + \mathbf{G}_c. \tag{5.3c}$$

The expressions for the force, $\mathbf{F}$, and torque, $\mathbf{G}_c$, in the body frame are derived in Appendix B as

$$\mathbf{F}(\mathbf{\Lambda}) = -\nabla_{\Lambda} V(\mathbf{\Lambda}) \tag{B.21}$$

$$\mathbf{G}_c(\mathbf{\Lambda}) = \mathbf{\Lambda}^{\times}\nabla_{\Lambda} V(\mathbf{\Lambda}) \tag{B.22}$$

where the potential is

$$V(\mathbf{\Lambda}) = -\int_{\mathcal{B}} \frac{G_*}{|\mathbf{\Lambda} + \mathbf{a}|} dm. \tag{B.11}$$

The system of equations in (5.3a) may be written as the noncanonical Hamiltonian system

$$\begin{pmatrix} \dot{\mathbf{\Sigma}} \\ \dot{\mathbf{\Lambda}} \\ \dot{\mathbf{\Pi}} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{1} & \mathbf{\Sigma}^{\times} \\ \mathbf{1} & \mathbf{0} & \mathbf{\Lambda}^{\times} \\ \mathbf{\Sigma}^{\times} & \mathbf{\Lambda}^{\times} & \mathbf{\Pi}^{\times} \end{bmatrix} \begin{pmatrix} \frac{1}{m}\mathbf{\Sigma} \\ \nabla_{\Lambda} V(\mathbf{\Lambda}) \\ \mathbf{I}^{-1}\mathbf{\Pi} \end{pmatrix} \tag{5.4}$$

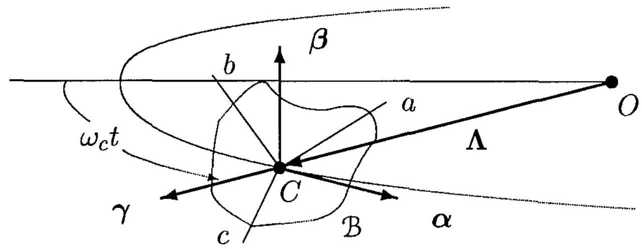


Figure 5.3 Configuration of Keplerian System

with Hamiltonian

$$H(\mathbf{z}) = \frac{1}{2}\mathbf{\Pi} \cdot \mathbf{I}^{-1}\mathbf{\Pi} + \frac{1}{2m}\mathbf{\Sigma} \cdot \mathbf{\Sigma} + V(\mathbf{\Lambda}). \tag{5.5}$$

This ninth-order system was first derived by Wang, *et al* [104]. The two first integrals for this reduced system are the Hamiltonian and the total angular momentum,

$$C_1(\mathbf{z}) = \frac{1}{2}\left|\mathbf{\Pi} + \mathbf{\Lambda}^{\times}\mathbf{\Sigma}\right|^2. \tag{5.6}$$

The latter arises as a Casimir function of the noncanonical system.

5.1.2 Keplerian System. In this problem we consider the motion of a rigid body about its center of mass, which is constrained to follow a circular Keplerian orbit about the center of attraction as shown in Figure 5.3. The frequency of a Keplerian orbit with radius $|\mathbf{\Lambda}|$ is

$$\omega_c = \left(\frac{G_*}{|\mathbf{\Lambda}|^3}\right)^{1/2}. \tag{5.7}$$

In the canonical treatment, the nature of the constraint leads to a derivation of the equations in an orbiting frame. The problem has three degrees of freedom associated with rotation about the center of mass and hence a sixth-order system results. The Hamiltonian for the canonical system is not the total energy, but it is an integral of motion. The canonical system has no other known integrals of motion for an arbitrary rigid body.

In deriving the noncanonical system, we will also be interested in motion relative to the orbital frame. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ be the body-frame representations of the orbital

frame unit vectors as shown in Figure 5.3. If $\mathbf{B}$ is the direction cosine matrix for the transformation from body to orbital frame, then $\mathbf{B}^{\mathsf{T}} = \begin{bmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{\gamma} \end{bmatrix}$. We then start by transforming Equation (5.1) to the body frame to get Euler's equation for the rotation of the body:

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi}^{\times}\mathbf{I}^{-1}\mathbf{\Pi} + \mathbf{G}_c = \mathbf{\Pi}^{\times}\mathbf{I}^{-1}\mathbf{\Pi} + \boldsymbol{\gamma}^{\times}\nabla_{\boldsymbol{\gamma}}V(\boldsymbol{\gamma}) \tag{5.8}$$

where we have used the expression for the torque given in Equation (B.23). The *relative* angular velocity and angular momentum may be defined as

$$\boldsymbol{\Omega}_r = \boldsymbol{\Omega} - \omega_c\boldsymbol{\beta} \tag{5.9a}$$

$$\mathbf{\Pi}_r = \mathbf{\Pi} - \omega_c\mathbf{I}\boldsymbol{\beta}. \tag{5.9b}$$

Differentiation of Equation (5.9b) leads to

$$\dot{\mathbf{\Pi}}_r = \mathbf{\Pi}_r{}^{\times}\mathbf{I}^{-1}\mathbf{\Pi}_r + \omega_c\mathbf{\Pi}_r{}^{\times}\boldsymbol{\beta} + \omega_c(\mathbf{I}\boldsymbol{\beta})^{\times}\mathbf{I}^{-1}\mathbf{\Pi}_r + \omega_c^2(\mathbf{I}\boldsymbol{\beta})^{\times}\boldsymbol{\beta} - \omega_c\mathbf{I}\dot{\boldsymbol{\beta}} + \boldsymbol{\gamma}^{\times}\nabla_{\boldsymbol{\gamma}}V(\boldsymbol{\gamma}). \tag{5.10}$$

In Appendix C, we show that $\boldsymbol{\beta}$ and $\mathbf{I}$ satisfy the identity

$$\boldsymbol{\beta}^{\times}\mathbf{I} + \mathbf{I}\boldsymbol{\beta}^{\times} \equiv \{[\mathrm{tr}\,(\mathbf{I})\,\mathbf{1} - \mathbf{I}]\,\boldsymbol{\beta}\}^{\times}. \tag{C.6}$$

We also have the rotational kinematic relations

$$\dot{\boldsymbol{\beta}} = \boldsymbol{\beta}^{\times}\mathbf{I}^{-1}\mathbf{\Pi}_r \tag{5.11a}$$

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}^{\times}\mathbf{I}^{-1}\mathbf{\Pi}_r. \tag{5.11b}$$

Using Equations (C.6) and (5.11a), Equation (5.10) reduces to

$$\dot{\mathbf{\Pi}}_r = \{\mathbf{\Pi}_r - \omega_c\,[\mathrm{tr}\,(\mathbf{I})\,\mathbf{1} - 2\mathbf{I}]\,\boldsymbol{\beta}\}^{\times}\mathbf{I}^{-1}\mathbf{\Pi}_r - \omega_c^2\boldsymbol{\beta}^{\times}\mathbf{I}\boldsymbol{\beta} + \boldsymbol{\gamma}^{\times}\nabla_{\boldsymbol{\gamma}}V(\boldsymbol{\gamma}). \tag{5.12}$$

Equations (5.11) and (5.12) form the noncanonical system

$$\begin{pmatrix} \dot{\mathbf{\Pi}}_r \\ \dot{\boldsymbol{\beta}} \\ \dot{\boldsymbol{\gamma}} \end{pmatrix} = \begin{bmatrix} \{\mathbf{\Pi}_r - \omega_c [\mathrm{tr}\,(\mathbf{I})\,\mathbf{1} - 2\mathbf{I}]\,\boldsymbol{\beta}\}^\times & \boldsymbol{\beta}^\times & \boldsymbol{\gamma}^\times \\ \boldsymbol{\beta}^\times & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\gamma}^\times & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{I}^{-1}\mathbf{\Pi}_r \\ -\omega_c^2 \mathbf{I}\boldsymbol{\beta} \\ \nabla_\gamma V(\boldsymbol{\gamma}) \end{pmatrix} \tag{5.13}$$

with Hamiltonian

$$H(\mathbf{z}) = \frac{1}{2}\mathbf{\Pi}_r \cdot \mathbf{I}^{-1}\mathbf{\Pi}_r - \frac{1}{2}\omega_c^2 \boldsymbol{\beta} \cdot \mathbf{I}\boldsymbol{\beta} + V(\boldsymbol{\gamma}). \tag{5.14}$$

This is the same Hamiltonian found in the canonical system and it is a first integral. The nullspace of the structure matrix is three-dimensional:

$$\mathcal{N}\,[\mathbf{J}(\mathbf{z})] = \mathrm{span} \left\{ \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\beta} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ \boldsymbol{\gamma} \\ \boldsymbol{\beta} \end{pmatrix} \right\}. \tag{5.15}$$

From the spanning vectors, we identify three independent Casimir functions:

$$C_1(\mathbf{z}) = \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\beta} \tag{5.16a}$$

$$C_2(\mathbf{z}) = \frac{1}{2}\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} \tag{5.16b}$$

$$C_3(\mathbf{z}) = \boldsymbol{\beta} \cdot \boldsymbol{\gamma}. \tag{5.16c}$$

In this problem, all three Casimir functions are trivial since $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are rows of the direction cosine matrix $\mathbf{B}$ so that

$$\boldsymbol{\beta} \cdot \boldsymbol{\beta} \equiv 1 \qquad \boldsymbol{\gamma} \cdot \boldsymbol{\gamma} \equiv 1 \qquad \boldsymbol{\beta} \cdot \boldsymbol{\gamma} \equiv 0. \tag{5.17}$$

We thus have a ninth-order system with four known first integrals.

5.1.3 Fixed-Point System. This system combines the simpler aspects of the two previous systems. Motion is described relative to the inertial frame as in the free rigid-body

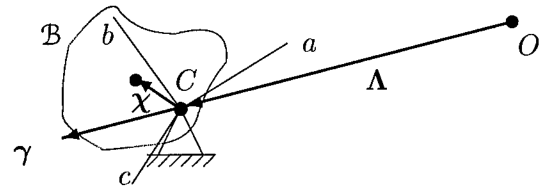


Figure 5.4 Configuration for Fixed-Point System

system, but there are only three degrees of freedom associated with rotation about the constrained point C as in the Keplerian system. The configuration is shown in Figure 5.4. In this problem, we do not assume C is the center of mass. However, we do assume C is fixed in inertial space so that the body angular momentum and torque are still related by

$$\dot{\boldsymbol{\pi}} = \mathbf{g}_c \tag{5.18}$$

in the inertial frame. In the body frame this becomes

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi}^{\times}\mathbf{I}^{-1}\mathbf{\Pi} + \mathbf{G}_c = \mathbf{\Pi}^{\times}\mathbf{I}^{-1}\mathbf{\Pi} + \boldsymbol{\gamma}^{\times}\nabla_{\gamma}V(\boldsymbol{\gamma}) \tag{5.19}$$

where we have again used Equation (B.23) for the torque. We also need the kinematic equation for the radial unit vector, $\boldsymbol{\gamma}$,

$$\dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}^{\times}\mathbf{I}^{-1}\mathbf{\Pi}. \tag{5.20}$$

These two equations form a noncanonical Hamiltonian system:

$$\begin{pmatrix} \dot{\mathbf{\Pi}} \\ \dot{\boldsymbol{\gamma}} \end{pmatrix} = \begin{bmatrix} \mathbf{\Pi}^{\times} & \boldsymbol{\gamma}^{\times} \\ \boldsymbol{\gamma}^{\times} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{I}^{-1}\mathbf{\Pi} \\ \nabla_{\gamma}V(\boldsymbol{\gamma}) \end{pmatrix} \tag{5.21}$$

with Hamiltonian

$$H(\mathbf{z}) = \frac{1}{2}\mathbf{\Pi} \cdot \mathbf{I}^{-1}\mathbf{\Pi} + V(\boldsymbol{\gamma}). \tag{5.22}$$

The structure matrix has a two-dimensional null space:

$$\mathcal{N}\left[\mathbf{J}(\mathbf{z})\right] = \operatorname{span}\left\{\begin{pmatrix}\gamma\\ \mathbf{\Pi}\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \gamma\end{pmatrix}\right\} \tag{5.23}$$

resulting in the Casimir functions

$$C_1(\mathbf{z}) = \mathbf{\Pi}\cdot\gamma \tag{5.24a}$$

$$C_2(\mathbf{z}) = \frac{1}{2}\gamma\cdot\gamma. \tag{5.24b}$$

The structure of this Hamiltonian system is identical to the heavy-top system given by Maddocks [62]. The only difference lies in the form of the potential.

5.2 Potential Approximations

In Appendix B, we present the series expansion of the gravitational potential and define the nth-order approximation as the potential which results when the expansion is truncated at $\mathcal{O}(\varepsilon^n)$. In the Hamiltonian systems presented above, we may directly substitute the nth-order potential approximation for the exact potential. The structure of the Hamiltonian system remains unchanged: the equations of motion and the Casimir functions given are equally valid for any potential. However, each choice of potential approximation results in a different Hamiltonian system. We refer to the system which results from substituting the nth-order potential approximation as the *nth-order model* of the Hamiltonian system.

5.3 Satellite Symmetry

The above derivations assume the satellite is an arbitrary rigid body. If the body has continuous rotational symmetry, further simplifications can be made. Continuous rotational symmetry implies either the mass distribution of the body is axisymmetric or it is spherically symmetric. If it is spherically symmetric, the attitude motion is a constant spin since the components of the angular momentum in the body frame are all integrals of motion. The translation of the satellite decouples from the attitude dynamics and we may

treat it as a point mass. If the body is axisymmetric, we have just one additional integral of motion — angular momentum about the symmetry axis. To treat this problem, the equations of motion are transformed from the body frame to a frame in which the body spins at a constant rate about the symmetry axis (along which we take one direction to be the third basis vector of the new frame). We refer to this new frame as the *nodal frame* since it turns out to be a generalization of the conventional nodal frame which arises as an intermediate frame when the transformation from orbital to body frame is parameterized by the classical Euler angles. The noncanonical Hamiltonian system in this new frame turns out to be identical to the Hamiltonian system for an arbitrary rigid body with one exception — the Hamiltonian has an additional term consisting of the new first integral multiplied by the spin rate of the nodal frame relative to the body frame. This result will be demonstrated for each axisymmetric system we treat.

5.4 The Hierarchy of Approximations

The hierarchy of Hamiltonian systems for a rigid body in a central gravitational field is shown in Figure 5.5. This is a $3 \times 3 \times \infty$ array of problems. The three layers associated with different levels of satellite symmetry are stacked in a column. The symmetry is shown in the upper left corner of each $3 \times \infty$ matrix. Within each matrix, the trajectory constraint increases as we move down and the order of the potential approximation decreases as we move to the right. Each element has two pairs of numbers. The pair on the left indicates the dimension and the number of known first integrals for the standard canonical system. The pair on the right indicates the dimension and number of first integrals for the noncanonical system. An asterisk in place of the first integral count indicates the system is known to be integrable. A bidirectional arrow connecting two matrix elements indicates the two systems are equivalent.

Before focusing on the specific problems to be addressed in this dissertation, we highlight a few points of interest. First, it is noteworthy that while the conversion from the canonical system to the noncanonical system for the free rigid-body system involves a

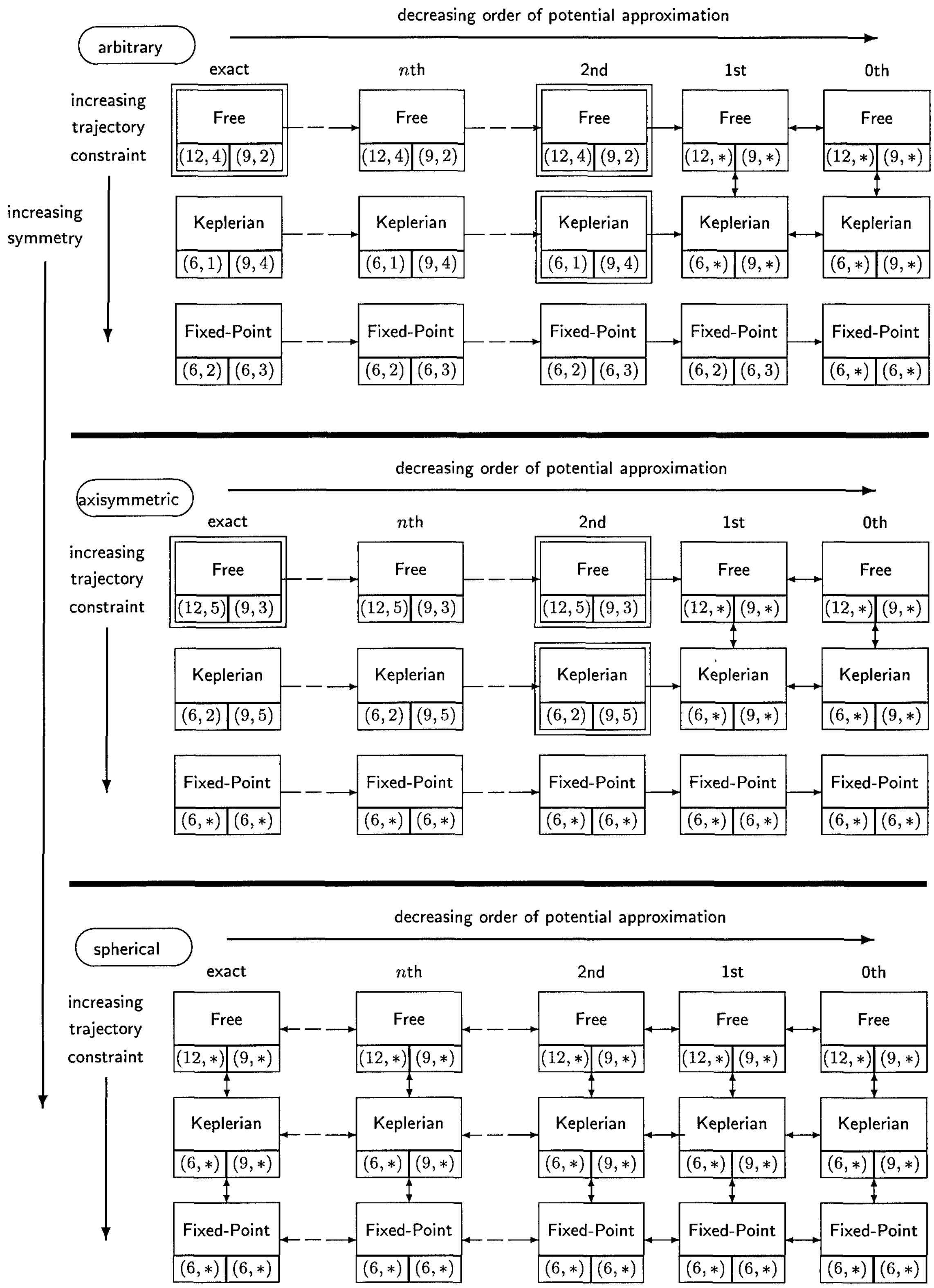


Figure 5.5 Hierarchy of Problems

reduction, the fixed-point problem does not change dimension and the Keplerian system actually increases dimension. It is also worthwhile to note the interplay between dimensions of the hierarchy. For instance, the order of the potential approximation determines how stringent the symmetry requirement is for the satellite. In particular, a second-order potential approximation includes only moment integrals up to second order. Therefore, the satellite is considered symmetric if it is dynamically symmetric, *i.e.*, it has equal moments of inertia. Also, in the free and Keplerian problems where we assume the origin of the body frame is the center of mass, the zeroth- and first-order potential approximations are identical (as shown in Appendix B).

Although we left the relationship between the center of mass and the origin of the body frame unspecified for the fixed-point problem, the symmetry requirements are with respect to the axes of the body frame. Hence, for the axisymmetric satellite, we may assume that the center of mass lies on the principal axis which is the axis of symmetry. For the spherically symmetric satellite, the origin must be the center of mass.

Beletskii [15] apparently recognized the significance of using different potential approximations, and discussion of the use of differing order approximations has appeared occasionally in the literature (for instance, Meirovitch [69] and Barkin [9]). Wang *et al* [104] were the first to address the topic of consistent approximations. They noted that the classical approximation of a rigid body in a central gravitational field (which is our second-order model of the Keplerian system) does not "[respect] the symmetries and conservation laws inherent in the problem". This is apparent in the derivations above. The Keplerian system conserves neither energy nor angular momentum whereas these quantities are both conserved in the two-body and free rigid-body systems.

This suggests that moving up and to the left in each matrix gives a better approximation of the two-body system. In the bottom matrix where the body is spherically symmetric, all systems are equivalent since in each case the body experiences no torque and spins at a constant rate. We therefore do not consider any problems from this matrix. We also omit further consideration of the fixed-point problem. Appendix 1 of [15]

provides a fairly complete treatment of this problem via classical means. Furthermore, portions of the presentation in the noncanonical treatment of the heavy-top problem by Maddocks [62] are also applicable (or can be easily extended) since the two problems have identical Hamiltonian structure.

Of the remaining systems in each of the upper two matrices, the three problems we investigate are highlighted by a double border. Beginning with the second-order model of the Keplerian system, we demonstrate that the classical results can be derived rather easily from the noncanonical formulation. We then consider the second order model of the free rigid-body system. This problem introduces orbital-attitude coupling neglected in the classical approximation. Finally, we address the exact model of the free rigid-body system. The last two problems were treated in part for an arbitrary body by Wang *et al* [104, 105]. We enhance their results for the arbitrary satellite and also consider the axisymmetric satellite.

VI. Keplerian System: Second-Order Approximation

In the following, we examine the relative equilibria of a rigid body which moves about a point (fixed in the body) which is constrained to follow a Keplerian orbit in a central gravitational field. By a Keplerian orbit, we mean the orbit of a particle with mass equal to that of the rigid body. For our purposes, it will suffice to consider the special case where the orbit is circular and the point following the orbit is the center of mass of the rigid body. We use the second-order approximation of the potential.

Classically this problem arises as an approximation to the motion of a satellite about a primary when the size of the satellite is small enough in relation to the radius of the orbit that the influence of attitude motion on the orbit may be neglected. The orbit is thus orthogonal with Keplerian frequency. We are concerned with motion relative to an orbiting frame as shown in Figure 5.3. We treat two cases: an arbitrary body and an axisymmetric body. The second-order potential approximation contains inertia integrals only up to second order so that the symmetry is in the dynamic sense.

The noncanonical formulation of the Keplerian system is a new development. However, the results regarding relative equilibria for the second-order model are well known. What is of interest in this chapter is the means by which we achieve those results. These same methods will be applied in the following chapters to the free rigid-body system for which considerably less analysis has been done previously.

6.1 Relative Equilibria for an Arbitrary Rigid Body

6.1.1 Nondimensional Hamiltonian System. In Appendix B, the second-order approximation of the potential is given as

$$V_2(\boldsymbol{\gamma}) = -\frac{G_* m}{|\boldsymbol{\Lambda}|} + \frac{G_* \boldsymbol{\gamma} \cdot \boldsymbol{\chi}}{|\boldsymbol{\Lambda}|^2} - \frac{1}{2}\frac{G_* \operatorname{tr}(\mathbf{I})}{|\boldsymbol{\Lambda}|^3} + \frac{3}{2}\frac{G_* \boldsymbol{\gamma} \cdot \mathbf{I}\boldsymbol{\gamma}}{|\boldsymbol{\Lambda}|^3}. \tag{B.36}$$

The first and third terms on the right-hand side may be neglected since they are constant for this problem. The second term also disappears since C is the center of mass and $\boldsymbol{\chi} = \mathbf{0}$.

Introducing the Keplerian frequency ω_c given in Equation (5.7), the remaining expression for the potential may be written as

$$V_2(\gamma) = \frac{3}{2}\omega_c^2 \gamma \cdot \mathbf{I}\gamma. \tag{6.1}$$

This potential approximation may be substituted into the Hamiltonian equations of motion for the Keplerian system given in Equation (5.13) to obtain

$$\begin{pmatrix} \dot{\bar{\mathbf{\Pi}}}_r \\ \dot{\boldsymbol{\beta}} \\ \dot{\boldsymbol{\gamma}} \end{pmatrix} = \begin{bmatrix} \left\{\bar{\mathbf{\Pi}}_r - \overline{\omega}_c \left[\mathrm{tr}\left(\bar{\mathbf{I}}\right)\mathbf{1} - 2\bar{\mathbf{I}}\right]\boldsymbol{\beta}\right\}^\times & \boldsymbol{\beta}^\times & \boldsymbol{\gamma}^\times \\ \boldsymbol{\beta}^\times & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\gamma}^\times & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \bar{\mathbf{I}}^{-1}\bar{\mathbf{\Pi}}_r \\ -\overline{\omega}_c^2 \bar{\mathbf{I}}\boldsymbol{\beta} \\ 3\overline{\omega}_c^2 \bar{\mathbf{I}}\boldsymbol{\gamma} \end{pmatrix}. \tag{6.2}$$

The overbar denotes dimensional variables. To nondimensionalize the system, we choose the mass, length, and time scales

$$m = \overline{m} = \int_{\mathcal{B}} d\overline{m} \qquad l = \left(\frac{\mathrm{tr}\left(\bar{\mathbf{I}}\right)}{\overline{m}}\right)^{\frac{1}{2}} \qquad t = \overline{\omega}_c^{-1}. \tag{6.3}$$

The equations of motion then become

$$\begin{pmatrix} \dot{\mathbf{\Pi}}_r \\ \dot{\boldsymbol{\beta}} \\ \dot{\boldsymbol{\gamma}} \end{pmatrix} = \begin{bmatrix} \left[\mathbf{\Pi}_r - \left(\mathbf{1} - 2\mathbf{I}\right)\boldsymbol{\beta}\right]^\times & \boldsymbol{\beta}^\times & \boldsymbol{\gamma}^\times \\ \boldsymbol{\beta}^\times & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\gamma}^\times & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{I}^{-1}\mathbf{\Pi}_r \\ -\mathbf{I}\boldsymbol{\beta} \\ 3\mathbf{I}\boldsymbol{\gamma} \end{pmatrix} \tag{6.4}$$

where $\mathbf{\Pi}_r = \left(m^{-1}l^{-2}t\right)\bar{\mathbf{\Pi}}_r$ and $\mathbf{I} = \left(m^{-1}l^{-2}\right)\bar{\mathbf{I}}$. This is the nondimensional second-order model of the Keplerian system. The Casimir functions for this system were given in Chapter 5 as

$$C_1(\mathbf{z}) = \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\beta} \tag{5.16a}$$
$$C_2(\mathbf{z}) = \frac{1}{2}\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} \tag{5.16b}$$
$$C_3(\mathbf{z}) = \boldsymbol{\beta} \cdot \boldsymbol{\gamma} \tag{5.16c}$$

and are already in dimensionless form.

6.1.2 Relative Equilibrium Conditions. We are interested in critical points of the Hamiltonian system (6.4) which may be characterized as described in Chapter 3 by the first-variation condition

$$\nabla F(\mathbf{z}_e) = \mathbf{0} \tag{3.9}$$

where for this system

$$F(\mathbf{z}) = H(\mathbf{z}) - \mu_1 C_1(\mathbf{z}) - \mu_2 C_2(\mathbf{z}) - \mu_3 C_3(\mathbf{z}). \tag{6.6}$$

The resulting relative equilibrium conditions are

$$\mathbf{I}^{-1}\mathbf{\Pi}_{r_e} = \mathbf{0} \tag{6.7a}$$

$$-\mathbf{I}\boldsymbol{\beta}_e - \mu_1\boldsymbol{\beta}_e - \mu_3\boldsymbol{\gamma}_e = \mathbf{0} \tag{6.7b}$$

$$3\mathbf{I}\boldsymbol{\gamma}_e - \mu_2\boldsymbol{\gamma}_e - \mu_3\boldsymbol{\beta}_e = \mathbf{0} \tag{6.7c}$$

along with the Casimir Conditions (5.16). Equation (6.7a) gives $\mathbf{\Pi}_{r_e} = \boldsymbol{\Omega}_{r_e} = \mathbf{0}$. Physically, this corresponds to the situation where the body is stationary with respect to the orbital frame. Taking the dot product of $\boldsymbol{\gamma}_e$ with Equation (6.7b) gives $\mu_3 = -\boldsymbol{\gamma}_e \cdot \mathbf{I}\boldsymbol{\beta}_e$ while the dot product of $\boldsymbol{\beta}_e$ with Equation (6.7c) gives $\mu_3 = 3\boldsymbol{\beta}_e \cdot \mathbf{I}\boldsymbol{\gamma}_e$. Symmetry of the inertia matrix implies these two requirements can only be satisfied if μ_3 is identically zero. Then the relative equilibrium conditions on $\boldsymbol{\beta}_e$ and $\boldsymbol{\gamma}_e$ become

$$\mathbf{I}\boldsymbol{\beta}_e = -\mu_1\boldsymbol{\beta}_e \tag{6.8a}$$

$$\mathbf{I}\boldsymbol{\gamma}_e = \frac{\mu_2}{3}\boldsymbol{\gamma}_e. \tag{6.8b}$$

We find that $\boldsymbol{\beta}_e$ and $\boldsymbol{\gamma}_e$ must be eigenvectors of the inertia matrix. Thus, the principal relative equilibria are solutions for this system, as first shown by Lagrange [52], and are in fact the only solutions — a result first demonstrated by Likins and Roberson [55].[1]

Without loss of generality, we choose the principal frame so that the relative equilibrium of interest is given as

$$\mathbf{\Pi}_{r_e} = \mathbf{0} \qquad \boldsymbol{\beta}_e = \mathbf{1}_2 \qquad \boldsymbol{\gamma}_e = \mathbf{1}_3 \tag{6.9a}$$

$$\mu_1 = -I_2 \qquad \mu_2 = 3I_3 \qquad \mu_3 = 0. \tag{6.9b}$$

Thus, at relative equilibrium the body frame and orbital frame are aligned.

6.1.3 Spectral Stability. Linearization of the noncanonical Hamiltonian system about the relative equilibrium gives $\dot{\delta\mathbf{z}} = \mathbf{A}(\mathbf{z}_e)\delta\mathbf{z}$ where $\delta\mathbf{z} = \mathbf{z} - \mathbf{z}_e$ and

$$\begin{aligned}
\mathbf{A}(\mathbf{z}) &= \mathbf{J}(\mathbf{z})\nabla^2 F(\mathbf{z}) \\
&= \begin{bmatrix} [\mathbf{\Pi}_r - (\mathbf{1} - 2\mathbf{I})\,\boldsymbol{\beta}]^\times & \boldsymbol{\beta}^\times & \boldsymbol{\gamma}^\times \\ \boldsymbol{\beta}^\times & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\gamma}^\times & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} - \mu_1\mathbf{1} & -\mu_3\mathbf{1} \\ \mathbf{0} & -\mu_3\mathbf{1} & 3\mathbf{I} - \mu_2\mathbf{1} \end{bmatrix} \\
&= \begin{bmatrix} [\mathbf{\Pi}_r - (\mathbf{1} - 2\mathbf{I})\,\boldsymbol{\beta}]^\times \mathbf{I}^{-1} & -\boldsymbol{\beta}^\times(\mathbf{I} + \mu_1\mathbf{1}) - \mu_3\boldsymbol{\gamma}^\times & -\mu_3\boldsymbol{\beta}^\times + \boldsymbol{\gamma}^\times(3\mathbf{I} - \mu_2\mathbf{1}) \\ \boldsymbol{\beta}^\times\mathbf{I}^{-1} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\gamma}^\times\mathbf{I}^{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix}.
\end{aligned} \tag{6.10}$$

[1]Our proof is more direct. Likins and Roberson began with the condition $3\boldsymbol{\gamma}_e{}^\times\mathbf{I}\boldsymbol{\gamma}_e - \boldsymbol{\beta}_e{}^\times\mathbf{I}\boldsymbol{\beta}_e = \mathbf{0}$ which is derived from the $\dot{\mathbf{\Pi}}_r$ equation with $\mathbf{\Pi}_r$ set equal to zero. We refer to this as the *Likins-Roberson Condition* for relative equilibrium. We will find that a similar condition arises for each system we consider.

Inserting the relative equilibrium conditions, the linear system matrix becomes

$$\mathbf{A}(\mathbf{z}_e) =$$

$$\begin{bmatrix} 0 & 0 & -(1-2I_2)/I_3 & 0 & 0 & I_2 - I_3 & 0 & -3(I_2 - I_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3(I_1 - I_3) & 0 & 0 \\ (1-2I_2)/I_1 & 0 & 0 & I_1 - I_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/I_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1/I_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.11}$$

We find that the system decouples into four subsystems: a pitch system associated with (Π_{r_2}, γ_1), a roll-yaw system associated with $(\Pi_{r_1}, \Pi_{r_3}, \beta_1, \beta_3, \gamma_2)$, and two trivial systems associated with β_2 and γ_3. The characteristic polynomial takes the form

$$P(s) = s^3 \left(s^2 + A_0\right) \left(s^4 + B_2 s^2 + B_0\right). \tag{6.12}$$

The three zero roots are associated with the three Casimir functions as discussed in Chapter 3. The coefficients of the polynomial are

$$A_0 = 3k_2 \qquad B_2 = 1 + 3k_1 + k_1 k_3 \qquad B_0 = 4k_1 k_3 \tag{6.13}$$

where we introduce the Smelt inertia parameters

$$k_1 = \frac{I_2 - I_3}{I_1} \qquad k_2 = \frac{I_1 - I_3}{I_2} \qquad k_3 = \frac{I_2 - I_1}{I_3}. \tag{6.14}$$

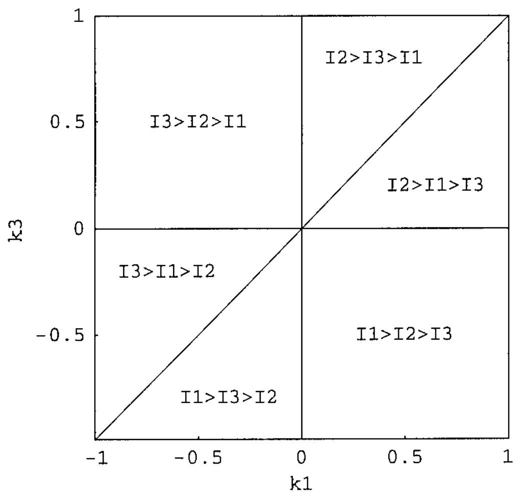


Figure 6.1 Regions of the Smelt Parameter Plane

Only two of these parameters are independent so that we may express k_2 as a function of k_1 and k_3:

$$k_2 = \frac{k_1 - k_3}{1 - k_1 k_3}. \tag{6.15}$$

In addition, these parameters satisfy the constraints $|k_i| < 1$. The six regions of the k_1–k_3 plane formed by the $k_1 = k_3$, $k_1 = 0$, and $k_3 = 0$ lines each correspond to different choices of the major and minor axis directions as shown in Figure 6.1. The spectral stability requirements from Table 3.1 are

$$A_0 \geq 0 \qquad B_0 \geq 0 \qquad B_2 \geq 0 \qquad B_2^2 - 4B_0 \geq 0 \tag{6.16}$$

which, upon eliminating strictly positive factors, reduces to

$$k_1 - k_3 \geq 0 \tag{6.17a}$$

$$1 + 3k_1 + k_1 k_3 \geq 0 \tag{6.17b}$$

$$k_1 k_3 \geq 0 \tag{6.17c}$$

$$(1 + 3k_1 + k_1 k_3)^2 - 16 k_1 k_3 \geq 0. \tag{6.17d}$$

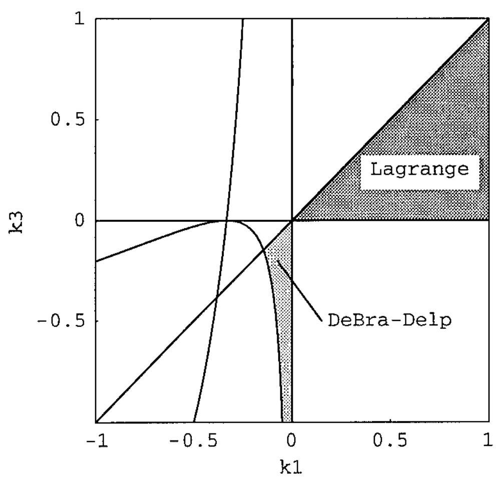


Figure 6.2 Stability Diagram for Tri-Inertial Keplerian System (medium gray = nonlinearly and spectrally stable, light gray = spectrally stable, white = unstable)

Condition (6.17a) implies that in the orbital plane the smaller principal inertia axis must be directed along the radial while Condition (6.17c) implies that the orbit normal cannot be the intermediate principal axis. The regions satisfying Conditions (6.17) are shown in the Smelt parameter plane in Figure 6.2. The dark gray region in the first quadrant corresponds to the stable relative equilibria identified by Lagrange [52]. The light gray region in the third quadrant was identified by DeBra and Delp [23]. The DeBra-Delp relative equilibria represent configurations for which the stabilizing effect of the rotation due to the orbital motion overcomes the destabilizing effect of the inertia ratios. The stability of these equilibria are an artifact of the rigid-body assumption. In the presence of energy dissipation, they are unstable [40].

It is important to recognize that, for a given tri-inertial rigid body, the twenty-four principal relative equilibria are grouped in sets of four[2] which lie at a single point in each of the six regions shown in Figure 6.1. Therefore, four relative equilibria *must* lie in the Lagrange region and an additional four *may* lie in the DeBra-Delp region. Thus, a given tri-inertial configuration has either four or eight spectrally stable relative equilibria.

[2]For a given principal relative equilibrium, three more with identical moments of inertia are found by rotating the body 180 degrees about one of the principal axes.

6.1.4 Nonlinear Stability. In this first problem, we will consider all four of the nonlinear stability analysis methods of Chapter 3 for comparison. We find that two of the methods fail for this system while the other two successfully demonstrate stability of the Lagrange region. No conclusion can be drawn by any of these methods regarding the DeBra-Delp region. Portions of the DeBra-Delp region have been shown to be nonlinearly stable by Modi and Marandi [65] through application of an extension of KAM theory to the normal form of Hamiltonian systems. Their method is beyond the scope of this dissertation and will not be explored here.

Direct Consideration of the Variational Lagrangian. In the process of computing the linear system matrix, we also computed the Hessian of the variational Lagrangian. From Equation (6.10), we have

$$\nabla^2 F(\mathbf{z}) = \begin{bmatrix} \mathbf{I}^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} - \mu_1 \mathbf{1} & -\mu_3 \mathbf{1} \\ \mathbf{0} & -\mu_3 \mathbf{1} & 3\mathbf{I} - \mu_2 \mathbf{1} \end{bmatrix}. \tag{6.18}$$

At relative equilibrium, this becomes

$$\nabla^2 F(\mathbf{z}_e) = \begin{bmatrix} 1/I_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 - I_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_2 - I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3(I_1 - I_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3(I_2 - I_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.19}$$

which is clearly not positive- or negative-definite. Thus, this method fails for this problem.

Projection Method. Although the previous results for the variational Lagrangian were inconclusive, Equation (6.19) is positive-semidefinite in the Lagrange region and suggestive that $H(\mathbf{z}_e)$ may be a constrained minimum. Following the procedure outlined in Section 3.3.2, we construct a matrix of basis vectors for $\mathcal{N}(\mathbf{J}(\mathbf{z}_e)) = \mathcal{N}\left(\mathbf{A}^\top(\mathbf{z}_e)\right)$ as $\mathbf{K}(\mathbf{z}_e) = \begin{bmatrix}\nabla C_1(\mathbf{z}_e) & \nabla C_2(\mathbf{z}_e) & \nabla C_3(\mathbf{z}_e)\end{bmatrix}$ and compute the projection onto the tangent space $TM|_{\mathbf{z}_e}$ as

$$\begin{aligned}\mathbf{P}(\mathbf{z}_e) &= \mathbf{1} - \mathbf{K}(\mathbf{z}_e)\left(\mathbf{K}^\top(\mathbf{z}_e)\mathbf{K}(\mathbf{z}_e)\right)^{-1}\mathbf{K}^\top(\mathbf{z}_e) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.\end{aligned} \tag{6.20}$$

From this we may compute the projected Hessian

$$\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e) = \begin{bmatrix} 1/I_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 - I_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_2 - I_3 & 0 & -(I_2 - I_3) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3(I_1 - I_3) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(I_2 - I_3) & 0 & I_2 - I_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.21}$$

for which we find that the eigenvalues are

$$\{0,\ 0,\ 0,\ 1/I_1,\ 1/I_2,\ 1/I_3,\ I_2 - I_1,\ 2(I_2 - I_3),\ 3(I_1 - I_3)\}\,.$$

The three zero eigenvalues are associated with the nullspace and the remaining five are associated with the tangent space of the constraint manifold. The conditions for positive-definiteness on the tangent space are precisely the conditions for the Lagrange region ($I_2 > I_1 > I_3$). That is, in this region the Hamiltonian is a constrained minimum and we may conclude nonlinear stability.

Energy-Casimir Method. The energy-Casimir method appears to offer a significant advantage in terms of flexibility of the choice of Liapunov function. Consider the function

$$H_\Phi(\mathbf{z}) = \Phi\left(H(\mathbf{z}), C_1(\mathbf{z}), C_2(\mathbf{z}), C_3(\mathbf{z})\right) \tag{6.22}$$

where $\Phi(x_0, x_1, x_2, x_3)$ is a smooth scalar function. The first variation evaluated at the relative equilibrium gives the condition

$$\nabla H_\Phi(\mathbf{z}_e) \;=\; \Phi_{x_0}\nabla H(\mathbf{z}_e) + \Phi_{x_1}\nabla C_1(\mathbf{z}_e) + \Phi_{x_2}\nabla C_2(\mathbf{z}_e) + \Phi_{x_3}\nabla C_3(\mathbf{z}_e) \;=\; \mathbf{0} \tag{6.23}$$

where Φ_{x_i} denotes the partial derivative of $\Phi(x_0, x_1, x_2, x_3)$ with respect to the ith argument, evaluated at the relative equilibrium. This reduces to give the conditions

$$\Phi_{x_1} = I_2\Phi_{x_0} \qquad \Phi_{x_2} = -3I_3\Phi_{x_0} \qquad \Phi_{x_3} = 0. \tag{6.24}$$

Computing the second variation and incorporating conditions (6.24), we find

$$
\nabla^2 H_\Phi(\mathbf{z}_e) =
\begin{bmatrix}
\frac{\Phi_{x0}}{I_1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\Phi_{x0}}{I_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\Phi_{x0}}{I_3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & (I_2 - I_1)\Phi_{x0} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Phi_{x_1,x_1} - 2I_2\Phi_{x_0,x_1} + I_2^2\Phi_{x_0,x_0} & \Phi_{x_1,x_3} - I_2\Phi_{x_0,x_3} & 0 & \Phi_{x_1,x_3} - I_2\Phi_{x_0,x_3} & \Phi_{x_1,x_2} - I_2\Phi_{x_0,x_2} + 3I_3\Phi_{x_0,x_1} - I_2\Phi_{x_0,x_0} \\
0 & 0 & 0 & 0 & \Phi_{x_1,x_3} - I_2\Phi_{x_0,x_3} & \Phi_{x_3,x_3} + (I_2 - I_3)\Phi_{x0} & 0 & \Phi_{x_3,x_3} & \Phi_{x_2,x_3} + 3I_3\Phi_{x_0,x_3} \\
0 & 0 & 0 & 0 & 0 & 0 & 3(I_1 - I_3)\Phi_{x0} & 0 & 0 \\
0 & 0 & 0 & 0 & \Phi_{x_1,x_3} - I_2\Phi_{x_0,x_3} & \Phi_{x_3,x_3} & 0 & \Phi_{x_3,x_3} + 3(I_2 - I_3)\Phi_{x0} & \Phi_{x_2,x_3} + 3I_3\Phi_{x_0,x_3} \\
0 & 0 & 0 & 0 & \Phi_{x_1,x_2} - I_2\Phi_{x_0,x_2} + 3I_3\Phi_{x_0,x_1} - I_2\Phi_{x_0,x_0} & \Phi_{x_2,x_3} + 3I_3\Phi_{x_0,x_3} & 0 & \Phi_{x_2,x_3} + 3I_3\Phi_{x_0,x_3} & \Phi_{x_2,x_2} + 6I_3\Phi_{x_0,x_2} + 9I_3^2\Phi_{x_0,x_0}
\end{bmatrix} . \quad (6.25)
$$

where Φ_{x_i,x_j} is the second partial derivative of $\Phi(x_0, x_1, x_2, x_3)$ with respect to the ith and jth arguments, evaluated at the relative equilibrium. For positive definiteness, the first three diagonal elements require Φ_{x_0} to be positive. Then consideration of the fourth and seventh diagonal elements assures that our results are limited to the Lagrange region. Without loss of generality, we may assume the Liapunov function is of the restricted class of functions $H_\phi(\mathbf{z}) = H(\mathbf{z}) + \sum_{i=1}^{m} \phi_i(C_i(\mathbf{z}))$ described in Section 3.3.2. Thus, $\Phi_{x_0} = 1$, $\Phi_{x_0,x_0} = 0$, and $\Phi_{x_i,x_j} = 0$ for $i \neq j$. The conditions for $\nabla^2 H_\Phi(\mathbf{z}_e)$ to be positive definite reduce to

$$I_2 > I_1 > I_3 \tag{6.26a}$$

$$\Phi_{x_1,x_1} > 0 \qquad \Phi_{x_2,x_2} > 0 \qquad \Phi_{x_3,x_3} > \frac{-3}{4}(I_2 - I_3). \tag{6.26b}$$

Let

$$\Phi(x_0, x_1, x_2, x_3) = x_0 + I_2 x_1^2 + 3I_3(1 - x_2)^2. \tag{6.27}$$

This function satisfies conditions (6.24) and (6.26). With the given function,

$$H_\Phi(\mathbf{z}) = \frac{1}{2}\mathbf{\Pi}_r \cdot \mathbf{I}^{-1}\mathbf{\Pi}_r - \frac{1}{2}\boldsymbol{\beta} \cdot \mathbf{I}\boldsymbol{\beta} + \frac{3}{2}\boldsymbol{\gamma} \cdot \mathbf{I}\boldsymbol{\gamma} + I_2\left(\frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{\beta}\right)^2 + 3I_3\left(1 - \frac{1}{2}\boldsymbol{\gamma} \cdot \boldsymbol{\gamma}\right)^2 \tag{6.28}$$

is a Liapunov function and the nonlinear stability of the Lagrange region is proven. The above Liapunov function should be contrasted with the classical Liapunov function used to prove the nonlinear stability of the Lagrange region,[3] which may be expressed in our variables as

$$v(\mathbf{z}) = H(\mathbf{z}) - H(\mathbf{z}_e) = \frac{1}{2}\mathbf{\Pi}_r \cdot \mathbf{I}^{-1}\mathbf{\Pi}_r - \frac{1}{2}\boldsymbol{\beta} \cdot (\mathbf{I} - I_2\mathbf{1})\boldsymbol{\beta} + \frac{3}{2}\boldsymbol{\gamma} \cdot (\mathbf{I} - I_3\mathbf{1})\boldsymbol{\gamma}. \tag{6.29}$$

[3] See, *e.g.*, Hughes [40:297].

This function is precisely the variational Lagrangian which we have previously shown to be only positive *semi*-definite in our choice of variables, which are dependent due to the Casimir constraints. In the classical treatment, the constraints are incorporated directly to eliminate dependent variables. This is effectively what we did above in the projection method.

Lagrange Multiplier Method. In this method, we attempt to determine a solution for the relative equilibria as a function of the Lagrange multipliers and then look at how this solution is embedded in the surface $F(\mathbf{z}_e(\boldsymbol{\mu}))$. Interestingly, while this method works very well for the closely related heavy top problem as considered by Maddocks [62], it breaks down completely for the current problem. To see why this happens, consider the relative equilibrium condition which, since $F(\mathbf{z}(\mu), \mu)$ is quadratic in the phase variables, may be written as

$$\nabla_z^2 F(\mu)\mathbf{z}_e = \mathbf{0}. \tag{6.30}$$

Unlike the heavy top problem, this system has no valid solutions when $\nabla_z^2 F(\mu)$ is nonsingular since that requires $\boldsymbol{\beta} = \boldsymbol{\gamma} = \mathbf{0}$, violating the Casimir constraints. Furthermore, the singularity condition $(I_i - \mu_1)(3I_i - \mu_2) + \mu_3^2 = 0$ must be satisfied for two of the three possible values of i in order for the Casimir constraints to be satisfied. This implies that for a tri-inertial configuration the singular solutions are isolated in μ-space. Hence, no relationship may be determined for the relative equilibrium as a function of the Lagrange multipliers. This should be clear since the relative equilibria are the principal relative equilibria regardless of inertias.

In point of fact, our problem is analogous to the Euler case of the heavy top problem since we have required the constrained point to be the center of mass. This limiting case is not treated by Maddocks [62]. If we remove this requirement, the left-hand side of Equation (6.30) would contain parameters describing the position of the center of mass

relative to the constrained point and we might expect the method to again be successful. Such an analysis is of limited interest here and we shall not pursue it.

6.2 Relative Equilibria for an Axisymmetric Rigid Body

6.2.1 Transformation to Nodal Frame. We now consider the Keplerian problem with the additional restriction that the body has an axis of symmetry. Dynamic symmetry is sufficient for the second-order approximation we treat here. Associated with the symmetry is an additional ignorable coordinate (and a conserved quantity). To treat this problem, the equations of motion are transformed from the body frame to a frame in which the body spins at a constant rate about the symmetry axis. We take one direction along the symmetry axis to be the third basis vector of both the body frame and the new frame. We refer to the new frame as the *nodal frame* since it turns out to be a generalization of the conventional nodal frame which arises as an intermediate frame when the transformation from orbital to body frame is parameterized by the classical Euler angles. The choice of the symmetry axis as a basis vector implies both the body and nodal frames are principal inertia frames.

The transformation to the nodal frame is given by

$$\mathbf{z} = \mathbf{M}\mathbf{z}_b \tag{6.31}$$

where $\mathbf{z}$ is the phase variable vector in the nodal frame, $\mathbf{z}_b$ is the phase variable vector in the body frame, and

$$\mathbf{M} = \begin{bmatrix} \mathbf{C}_{nb} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{nb} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{nb} \end{bmatrix} \tag{6.32}$$

with

$$\mathbf{C}_{nb} = \begin{bmatrix} \cos\mu_4 t & -\sin\mu_4 t & 0 \\ \sin\mu_4 t & \cos\mu_4 t & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6.33}$$

Here μ_4 is the rotation rate of the body about the symmetry axis relative to the nodal frame. The new conserved quantity is the angular momentum component in the direction of the symmetry axis, which is given by

$$C_4(\mathbf{z}) = \mathbf{\Pi} \cdot \mathbf{1}_3 = (\mathbf{\Pi}_r + \mathbf{I}\boldsymbol{\beta}) \cdot \mathbf{1}_3. \tag{6.34}$$

We will now show that the Hamiltonian form of the system remains unchanged with one exception — the Hamiltonian takes the form

$$H(\mathbf{z}) = H_b(\mathbf{z}) - \mu_4 C_4(\mathbf{z}) \tag{6.35}$$

where $H_b(\mathbf{z})$ is the previous Hamiltonian expressed in terms of the new phase variables. Differentiating Equation (6.31) with respect to time gives

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{M}\dot{\mathbf{z}}_b + \dot{\mathbf{M}}\mathbf{z}_b \\ &= \mathbf{M}\mathbf{J}(\mathbf{z}_b)\nabla H_b(\mathbf{z}_b) + \dot{\mathbf{M}}\mathbf{M}^{\mathsf{T}}\mathbf{z} \\ &= \mathbf{M}\mathbf{M}^{\mathsf{T}}\mathbf{J}(\mathbf{z})\mathbf{M}\mathbf{M}^{\mathsf{T}}\nabla H_b(\mathbf{z}) + \mathbf{W}\mathbf{z} \\ &= \mathbf{J}(\mathbf{z})\nabla H_b(\mathbf{z}) + \mathbf{W}\mathbf{z} \end{aligned} \tag{6.36}$$

where

$$\mathbf{W} = \begin{bmatrix} \dot{\mathbf{C}}_{nb}\mathbf{C}_{nb}^{\mathsf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dot{\mathbf{C}}_{nb}\mathbf{C}_{nb}^{\mathsf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dot{\mathbf{C}}_{nb}\mathbf{C}_{nb}^{\mathsf{T}} \end{bmatrix} = \begin{bmatrix} \mu_4\mathbf{1}_3{}^{\times} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mu_4\mathbf{1}_3{}^{\times} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mu_4\mathbf{1}_3{}^{\times} \end{bmatrix}. \tag{6.37}$$

We may show that $\mathbf{Wz} = -\mu_4 \mathbf{J}(\mathbf{z})\nabla C_4(\mathbf{z})$ by direct computation so that

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{J}(\mathbf{z})\left[\nabla H_b(\mathbf{z}) - \mu_4 \nabla C_4(\mathbf{z})\right] \\ &= \mathbf{J}(\mathbf{z})\nabla H(\mathbf{z}) \end{aligned} \tag{6.38}$$

as claimed.

6.2.2 Relative Equilibrium Conditions. Given the transformation above, the relative equilibria of the axisymmetric system are now characterized by critical points of

$$F(\mathbf{z}) = H(\mathbf{z}) - \sum_{i=1}^{3} \mu_i C_i(\mathbf{z}) = H_b(\mathbf{z}) - \sum_{i=1}^{4} \mu_i C_i(\mathbf{z}). \tag{6.39}$$

Taking the gradient, we find the relative equilibrium conditions

$$\mathbf{I}^{-1}\mathbf{\Pi}_{r_e} - \mu_4 \mathbf{1}_3 = \mathbf{0} \tag{6.40a}$$

$$-\mathbf{I}\boldsymbol{\beta}_e - \mu_1 \boldsymbol{\beta}_e - \mu_3 \boldsymbol{\gamma}_e - \mu_4 \mathbf{I}\mathbf{1}_3 = \mathbf{0} \tag{6.40b}$$

$$3\mathbf{I}\boldsymbol{\gamma}_e - \mu_2 \boldsymbol{\gamma}_e - \mu_3 \boldsymbol{\beta}_e = \mathbf{0}. \tag{6.40c}$$

Along with the Casimir constraints (5.16) and the symmetry integral (6.34), these define a family of relative equilibria parameterized by μ_4 and the inertias. Equation (6.40a) gives $\boldsymbol{\Omega}_{r_e} = \mu_4 \mathbf{1}_3$ which implies at relative equilibrium the body rotates with angular velocity μ_4 about the symmetry axis relative to the orbital frame (*i.e.*, the nodal frame is fixed with respect to the orbital frame). We eliminate the Lagrange multipliers μ_1, μ_2, and μ_3 from Equations (6.40b) and (6.40c) by taking the crossproduct of $\boldsymbol{\beta}_e$ with the first and $\boldsymbol{\gamma}_e$ with the second and adding to get

$$3\boldsymbol{\gamma}_e^{\times}\mathbf{I}\boldsymbol{\gamma}_e - \boldsymbol{\beta}_e^{\times}\mathbf{I}\boldsymbol{\beta}_e - \mu_4 \boldsymbol{\beta}_e^{\times}\mathbf{I}\mathbf{1}_3 = \mathbf{0}. \tag{6.41}$$

We refer to this as the *Pringle-Likins Condition* for axisymmetric relative equilibria since it is a generalization of the relative equilibrium conditions given by Pringle [81] and Likins [56] for specific choices of Euler angles. Note the similarity to the Likins-Roberson Condition

found for tri-inertial bodies. This condition is applicable for any parameterization of the transformation from orbital to nodal frame. Let $I_1 = I_2 = I_t$ and $I_3 = I_a$ be the transverse and axial principal moments of inertia. Then Equation (6.41) becomes

$$\begin{pmatrix} (I_a - I_t)(3\gamma_2\gamma_3 - \beta_2\beta_3) - \mu_4 I_a \beta_2 \\ (I_a - I_t)(3\gamma_1\gamma_3 - \beta_1\beta_3) - \mu_4 I_a \beta_1 \\ 0 \end{pmatrix} = \mathbf{0}. \tag{6.42}$$

Let the inertia parameter k_t and the spin parameter Q be given by

$$k_t = (I_a - I_t)/I_t \tag{6.43a}$$

$$Q = -(I_a/I_t)\mu_4 = -(1 + k_t)\mu_4. \tag{6.43b}$$

Then, multiplying Equation (6.42) by $1/I_t$, we may write the scalar form of the Pringle-Likins Condition as

$$3k_t\gamma_i\gamma_3 - k_t\beta_i\beta_3 + Q\beta_i = 0 \quad (i = 1, 2). \tag{6.44}$$

There are three classes of solutions to these conditions:[4]

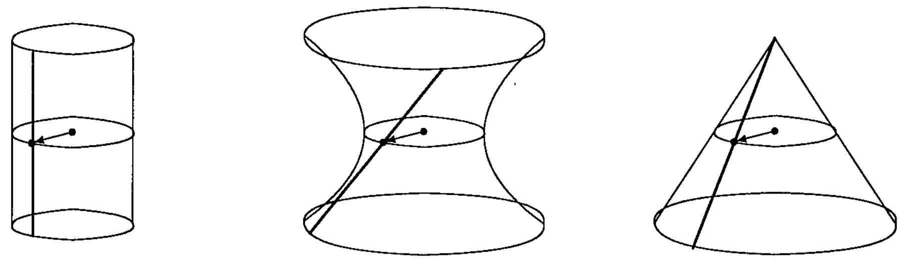
i. Cylindrical Relative Equilibria: $\beta_3^2 = 1$

ii. Hyperbolic Relative Equilibria: $\gamma_3 = 0 \quad \beta_3 = Q/k_t \quad (k_t \neq 0)$

iii. Conical Relative Equilibria: $\alpha_3 = 0 \quad \beta_3 = Q/(4k_t) \quad (k_t \neq 0)$.

These relative equilibria were shown in Figure 1.3 which we repeat here as Figure 6.3 for convenience. To be complete, we note that the orbital frame unit vector $\boldsymbol{\alpha}$ is defined in terms of phase variables by $\boldsymbol{\alpha} = \boldsymbol{\beta}^{\times}\boldsymbol{\gamma}$. Only the components α_3, β_3, and γ_3 along the symmetry axis are determinate. This situation occurs because there are infinitely many equally valid choices for the nodal frame. Given one choice, any rotation about the

[4] When $k_t = 0$ only the cylindrical relative equilibria are possible for a nonzero spin rate. Equality of the axial and transverse inertias results in the attitude motion of the body being decoupled from the orbit. The angular velocity vector is fixed in inertial space and, while the vector may point in any direction, only when the vector points normal to the orbit is the motion a relative equilibrium.



(a) Cylindrical (b) Hyperbolic (c) Conical

Figure 6.3 Relative Equilibrium Classes for Axisymmetric Satellites (after Likins [56])

symmetry axis by a fixed amount gives another. We will introduce additional constraints to completely specify the nodal frame and simplify our computations; however, this is not a necessary step since we are only interested in the orientation of the symmetry axis in the orbital frame. Let $\boldsymbol{\eta}$ denote the unit vector along this axis so that $\boldsymbol{\eta} = \mathbf{B}\mathbf{1}_3 = [\boldsymbol{\alpha}\,\boldsymbol{\beta}\,\boldsymbol{\gamma}]^{\mathsf{T}}\,\mathbf{1}_3 = (\alpha_3, \beta_3, \gamma_3)$. Hence, this vector is completely specified for each class of solution and the physical orientations shown in Figure 6.3 are readily apparent.

To derive these solutions, we write Equation (6.44) as the linear system

$$\begin{bmatrix} -k_t\beta_1 & 3k_t\gamma_1 \\ -k_t\beta_2 & 3k_t\gamma_2 \end{bmatrix} \begin{pmatrix} \beta_3 \\ \gamma_3 \end{pmatrix} = \begin{pmatrix} -Q\beta_1 \\ -Q\beta_2 \end{pmatrix}. \tag{6.45}$$

The coefficient matrix is nonsingular for $k_t^2\beta_1\gamma_2 \neq k_t^2\beta_2\gamma_1$ (or $k_t^2\alpha_3 \neq 0$). Inverting this matrix, we find

$$\begin{pmatrix} \beta_3 \\ \gamma_3 \end{pmatrix} = \frac{1}{-3k_t^2\alpha_3} \begin{bmatrix} 3k_t\gamma_2 & -3k_t\gamma_1 \\ k_t\beta_2 & -k_t\beta_1 \end{bmatrix} \begin{pmatrix} -Q\beta_1 \\ -Q\beta_2 \end{pmatrix} = \begin{pmatrix} Q/k_t \\ 0 \end{pmatrix}. \tag{6.46}$$

This is the hyperbolic solution of Likins [56].

Next consider the singular case where we have

$$k_t^2\alpha_3 = k_t^2\beta_1\gamma_2 - k_t^2\beta_2\gamma_1 = 0. \tag{6.47}$$

In addition we have the orthogonality constraint $\boldsymbol{\beta} \cdot \boldsymbol{\gamma} = \mathbf{0}$ which may be written as

$$\beta_1\gamma_1 + \beta_2\gamma_2 = -\beta_3\gamma_3. \tag{6.48}$$

We may then form another linear system

$$\begin{bmatrix} \beta_1 & \beta_2 \\ -k_t^2\beta_2 & k_t^2\beta_1 \end{bmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -\beta_3\gamma_3 \\ 0 \end{pmatrix}. \tag{6.49}$$

The coefficient matrix in this system is nonsingular when $k_t^2(\beta_1^2 + \beta_2^2) \neq 0$ (or, for $k_t \neq 0$, $\beta_3^2 \neq 1$). In this case, we may invert the matrix to get

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \frac{1}{k_t^2(\beta_1^2 + \beta_2^2)} \begin{bmatrix} k_t^2\beta_1 & -\beta_2 \\ k_t^2\beta_2 & \beta_1 \end{bmatrix} \begin{pmatrix} -\beta_3\gamma_3 \\ 0 \end{pmatrix} = \frac{-\beta_3}{\gamma_3} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \tag{6.50}$$

where we have recognized that $\beta_1^2 + \beta_2^2 = 1 - \beta_3^2 = \gamma_3^2$ (since $\alpha_3 = 0$ by the first singularity condition). This result may be substituted into Likins' condition which then reduces to

$$\beta_3 = Q/(4k_t). \tag{6.51}$$

This is the conical case of Likins [56].

Considering the singular case of system (6.49), we have $\beta_3^2 = 1$ which (recognizing that we must also have $\beta_1 = \beta_2 = \gamma_3 = 0$) satisfies the Pringle-Likins Condition directly. This is the cylindrical case of Likins [56].

We now return to the Lagrange multipliers which we previously eliminated. Taking the dot product of Equations (6.40b) and (6.40c) with $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, respectively, we find

$$\mu_1 = -\boldsymbol{\beta}_e \cdot \mathbf{I}\boldsymbol{\beta}_e - \mu_4\boldsymbol{\beta}_e \cdot \mathbf{I}\mathbf{1}_3 \tag{6.52a}$$

$$\mu_2 = 3\boldsymbol{\gamma}_e \cdot \mathbf{I}\boldsymbol{\gamma}_e. \tag{6.52b}$$

Conversely, taking the dot product of each of Equations (6.40b) and (6.40c) with $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$, respectively, we find

$$\mu_3 = -\boldsymbol{\gamma}_e \cdot \mathbf{I}\boldsymbol{\beta}_e - \mu_4 \boldsymbol{\gamma}_e \cdot \mathbf{I}\mathbf{1}_3 \tag{6.53a}$$

$$\mu_3 = 3\boldsymbol{\beta}_e \cdot \mathbf{I}\boldsymbol{\gamma}_e. \tag{6.53b}$$

Either one of these equations will serve our purpose.[5] Reducing Equations (6.52a), (6.52b), and (6.53b) to scalar form and expressing in terms of the inertias and spin rate, we obtain

$$\mu_1 = -\left[I_t + (I_a - I_t)\beta_3^2 + \mu_4 I_a \beta_3\right] \tag{6.54a}$$

$$\mu_2 = 3\left[I_t + (I_a - I_t)\gamma_3^2\right] \tag{6.54b}$$

$$\mu_3 = 3(I_a - I_t)\beta_3\gamma_3. \tag{6.54c}$$

In terms of the inertia and spin parameters defined above, these become

$$\mu_1 = -\frac{1 + k_t\beta_3^2 - Q\beta_3}{3 + k_t} \tag{6.55a}$$

$$\mu_2 = \frac{3\left(1 + k_t\gamma_3^2\right)}{3 + k_t} \tag{6.55b}$$

$$\mu_3 = \frac{3k_t\beta_3\gamma_3}{3 + k_t}. \tag{6.55c}$$

These expressions are valid for all three relative equilibrium classes and will be used in the stability analysis to follow.

The additional integral for the axisymmetric system given in Equation (6.34) is linear in the phase variables and has no influence on the Hessian of the variational Lagrangian $F(\mathbf{z})$ given in Equation (6.18). The general form of the linear system matrix shown in Equation (6.10) also remains valid for the axisymmetric system (recognizing that the variables are now expressed in the nodal frame rather than the body frame). However, inserting the relative equilibrium conditions results in different matrices for each class of relative equi-

[5] In fact, setting the right-hand sides equal is an alternative means of deriving the relative equilibrium conditions which more closely follows the method used earlier for the arbitrary rigid body.

libria. We treat the stability of the cylindrical, hyperbolic, and conical relative equilibria in turn.

6.2.3 *Cylindrical Relative Equilibria.*

Spectral Stability. For the cylindrical relative equilibria, the symmetry axis is aligned with the orbit normal so that $\beta_3^2 = 1$. Without loss of generality, let $\beta_3 = 1$. By the orthogonality of the unit vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$, this further implies $\alpha_3 = \gamma_3 = \beta_1 = \beta_2 = 0$. The choice of γ_1 and γ_2 is arbitrary as long as $\gamma_1^2 + \gamma_2^2 = 1$. For definiteness, we choose $\gamma_1 = 1$ and $\gamma_2 = 0$. The complete set of relative equilibrium conditions are then given as

$$\Pi_{r_1} = 0 \qquad \Pi_{r_2} = 0 \qquad \Pi_{r_3} = \mu_4 I_a \tag{6.56a}$$

$$\beta_1 = 0 \qquad \beta_2 = 0 \qquad \beta_3 = 1 \tag{6.56b}$$

$$\gamma_1 = 1 \qquad \gamma_2 = 0 \qquad \gamma_3 = 0 \tag{6.56c}$$

$$\mu_1 = -I_a(1+\mu_4) \qquad \mu_2 = 3I_t \qquad \mu_3 = 0. \tag{6.56d}$$

For these values, the linear system matrix becomes

$$\mathbf{A}(\mathbf{z}_e) = \begin{bmatrix} 0 & 1-k_t+Q & 0 & 0 & -\frac{kt-Q}{3+k_t} & 0 & 0 & 0 & 0 \\ -1+k_t-Q & 0 & 0 & \frac{kt-Q}{3+k_t} & 0 & 0 & 0 & 0 & \frac{-3k_t}{3+k_t} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -(3+k_t) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3+k_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3+k_t}{1+k_t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3+k_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{6.57}$$

where k_t and Q are the inertia and spin parameters defined in Equation (6.43). This system matrix decouples into a trivial pitch subsystem associated with $(\Pi_{r_3}, \beta_3, \gamma_1, \gamma_2)$ and a roll-yaw subsystem associated with $(\Pi_{r_1}, \Pi_{r_2}, \beta_1, \beta_2, \gamma_3)$. The characteristic equation takes the

form

$$P(s) = s^5(s^4 + A_2 s^2 + A_0). \tag{6.58}$$

We note that the pitch subsystem has all zero roots since its submatrix is lower triangular. This might be anticipated since the symmetry axis is aligned with the pitch axis. However, another way to consider this is that in addition to the three zero roots associated with the Casimir functions, two more roots are forced to zero by the symmetry of the body as discussed in Chapter 3. So we will find that the characteristic polynomial takes this form for the other two classes of relative equilibria as well, even though we will no longer be able to distinguish pitch and roll-yaw subsystems. The coefficients of the characteristic equation may be expressed as

$$A_0 = (k_t - Q)(4k_t - Q) \qquad A_2 = 1 + 3k_t + (k_t - Q)^2. \tag{6.59}$$

The spectral stability requirements from Table 3.1 are

$$A_0 \geq 0 \qquad A_2 \geq 0 \qquad A_2^2 - 4A_0 \geq 0 \tag{6.60}$$

which reduce to

$$(k_t - Q)(4k_t - Q) \geq 0 \tag{6.61a}$$

$$1 + 3k_t + (k_t - Q)^2 \geq 0 \tag{6.61b}$$

$$(k_t - Q)^2(4k_t - Q)^2 - 4(k_t - Q)^2 - 4(1 + 3k_t) \geq 0. \tag{6.61c}$$

The stability regions are shown in Figure 6.4 which plots the inertia parameter, k_t, versus the spin rate, μ_4. Shaded regions are stable. Only slow spin rates are shown ($|\mu_4| \leq 5$). The vertical axis $k_t = 0$ splits the parameter space into two regions with prolate configurations on the left and oblate configurations on the right. The horizontal axis $\mu_4 = 0$ represents configurations which are stationary in the orbital frame while the horizontal axis $\mu_4 = -1$

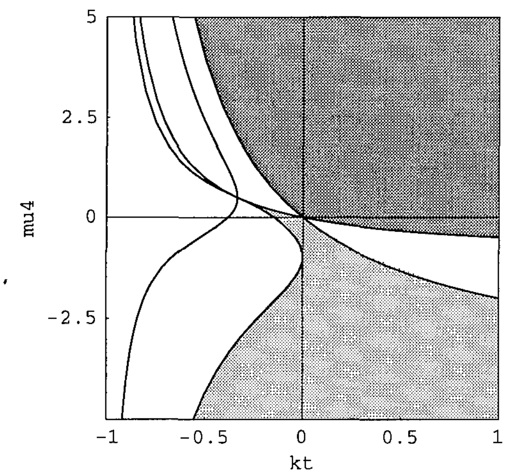


Figure 6.4 Stability Diagram for Cylindrical Relative Equilibria of Axisymmetric Keplerian System (medium gray = nonlinearly and spectrally stable, light gray = spectrally stable, white = unstable)

gives configurations which do not rotate in inertial space.[6] At larger values of $|\mu_4|$, the curves in the left half plane tend asymptotically toward $k_t = -1$. Thus, all but the very prolate configurations are spectrally stable.

Nonlinear Stability. For the remaining nonlinear stability analyses, we will focus on the projection method. Similar results should be obtained by the energy-Casimir method. The projection method as described in Chapter 3 assumed the system had no symmetries. The presence of a symmetry introduces a new first integral (additional constraint) and requires some adjustments to our approach.

Recall that in the projection method we construct the projection onto the tangent space at relative equilibrium (*i.e.*, onto $\mathcal{R}(\mathbf{A}(\mathbf{z}_e))$, the range of $\mathbf{A}(\mathbf{z}_e)$). For the arbitrary rigid body, $\mathcal{R}(\mathbf{A}(\mathbf{z}_e))$ coincided with $\mathcal{R}(\mathbf{J}(\mathbf{z}_e))$ as long as no two principal inertias were equal.[7] Recognizing that $\mathcal{N}(\mathbf{J}(\mathbf{z}_e))$, the nullspace of $\mathbf{J}(\mathbf{z}_e)$, is the orthogonal complement to $\mathcal{R}(\mathbf{J}(\mathbf{z}_e))$ (and thus $\mathcal{R}(\mathbf{A}(\mathbf{z}_e))$), a projection onto the tangent space is constructed by

[6]This latter characterization is only valid for the cylindrical relative equilibria. Relative equilibria with no inertial rotation are not possible for the other two classes.

[7]In terms of the characteristic polynomial (6.12), we require that A_0 and B_0 not equal zero so that no additional zero roots are present. These conditions are satisfied if and only if no two principal inertias are equal.

subtracting a projection onto $\mathcal{N}(\mathbf{J}(\mathbf{z}_e))$ from the identity matrix. The projection onto $\mathcal{N}(\mathbf{J}(\mathbf{z}_e))$ is easily constructed using the gradients of the Casimir functions as a basis.

With the addition of axial symmetry, $\mathcal{R}(\mathbf{A}(\mathbf{z}_e))$ no longer coincides with $\mathcal{R}(\mathbf{J}(\mathbf{z}_e))$, or conversely, $\mathcal{N}\left(\mathbf{A}^\mathsf{T}(\mathbf{z}_e)\right)$ no longer coincides with $\mathcal{N}(\mathbf{J}(\mathbf{z}_e))$. Two additional zero eigenvalues of $\mathbf{A}(\mathbf{z}_e)$ appear; however, the nullspace of $\mathbf{A}^\mathsf{T}(\mathbf{z}_e)$ is expanded by only one dimension spanned by the gradient of the first integral associated with the symmetry.[8] That is, the algebraic multiplicity of the zero eigenvalue is $m+2$ but the geometric multiplicity is only $m+1$ and

$$\nabla C_{m+1}(\mathbf{z}_e) \cdot \mathbf{J}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e) = -\mathbf{J}(\mathbf{z}_e)\nabla C_{m+1}(\mathbf{z}_e) \cdot \nabla^2 F(\mathbf{z}_e) = \mathbf{0} \tag{6.62}$$

However, for the axial symmetry treated here, we have

$$\mathbf{J}(\mathbf{z})\nabla C_{m+1}(\mathbf{z}) = -(1/\mu_{m+1})\mathbf{W}\mathbf{z} \tag{6.63}$$

where $\mathbf{W}$ is the matrix defined in Equation (6.37). Let

$$\boldsymbol{\xi}(\mathbf{z}) = -(1/\mu_{m+1})\mathbf{W}\mathbf{z} = \left(\mathbf{1}_3{}^\times\boldsymbol{\Pi}_r, \mathbf{1}_3{}^\times\boldsymbol{\beta}, \mathbf{1}_3{}^\times\boldsymbol{\gamma}\right). \tag{6.64}$$

Then we find

$$\boldsymbol{\xi}(\mathbf{z}_e) \cdot \nabla^2 F(\mathbf{z}_e) = \mathbf{0}. \tag{6.65}$$

Perturbations in the direction of $\boldsymbol{\xi}(\mathbf{z}_e)$ represent rotations about the symmetry axis and clearly are not prevented by the constraints. Thus, we find that the variational Lagrangian for axisymmetric systems must have at least a one-dimensional nullspace in the tangent

[8]While the gradients of the Casimir functions always lie in the nullspace of $\mathbf{A}^\mathsf{T}(\mathbf{z})$, this is only true for the gradient of the symmetry integral at relative equilibrium (and only if the symmetry integral is linear in the phase variables). To see this, consider that $dC_{m+1}(\mathbf{z})/dt = \nabla C_{m+1}(\mathbf{z}) \cdot \mathbf{J}(\mathbf{z})\nabla H(\mathbf{z}) = 0$. Taking the gradient, we find $\nabla^2 C_{m+1}(\mathbf{z}) \cdot \mathbf{J}(\mathbf{z})\nabla H(\mathbf{z}) + \nabla C_{m+1}(\mathbf{z}) \cdot \nabla\mathbf{J}(\mathbf{z})\nabla H(\mathbf{z}) + \nabla C_{m+1}(\mathbf{z}) \cdot \mathbf{J}(\mathbf{z})\nabla^2 H(\mathbf{z}) = \mathbf{0}$. The first term is identically $\mathbf{0}$ since the symmetry integral is linear in $\mathbf{z}$. By Equation (3.14), at relative equilibrium we may combine the remaining two terms to conclude that $\nabla C_{m+1}(\mathbf{z}_e) \cdot \mathbf{J}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e) = \mathbf{0}$.

space. This is related to the arbitrariness of the nodal frame and the concept of ignorable or cyclic coordinates in canonical systems. To treat this, we revise our definition of stability to omit perturbations in this direction.[9] Thus, in constructing the matrix of basis vectors for the constraints, we include $\boldsymbol{\xi}(\mathbf{z}_e)$. Note that we make no claims to the generality of this approach to systems with symmetry beyond the axisymmetric systems treated here.

For the current problem, the variational Lagrangian was given previously as

$$F(\mathbf{z}) = H(\mathbf{z}) - \sum_{i=1}^{3} \mu_i C_i(\mathbf{z}) = H_b(\mathbf{z}) - \sum_{i=1}^{4} \mu_i C_i(\mathbf{z}). \tag{6.39}$$

Note that in a variational sense, we are no longer seeking a constrained minimum of the Hamiltonian, but of the function $H_b(\mathbf{z})$ (which is the Hamiltonian of the arbitrary body problem expressed in nodal frame variables). The form of the Hessian of the variational Lagrangian given in Equation (6.18) is still applicable for the axisymmetric system because the additional integral is linear in the phase variables. Substituting the current relative equilibrium conditions, we have

$$\nabla^2 F(\mathbf{z}_e) = \operatorname{diag}\left(3+k_t \;\; 3+k_t \;\; \frac{3+k_t}{1+k_t} \;\; \frac{k_t - Q}{3+k_t} \;\; \frac{k_t - Q}{3+k_t} \;\; \frac{-Q}{3+k_t} \;\; 0 \;\; 0 \;\; \frac{3k_t}{3+k_t} \right) \tag{6.66}$$

which is clearly not positive definite, and is positive semi-definite only when $k_t > 0$ and $Q < 0$.

To construct the projection onto the portion of the tangent space orthogonal to the cyclic perturbation direction, we form the matrix of basis vectors for the $\mathcal{N}(\mathbf{A}(\mathbf{z}_e))$ augmented with the cyclic perturbation vector:

$$\mathbf{K}(\mathbf{z}_e) = \left[\nabla C_1(\mathbf{z}_e) \quad \nabla C_2(\mathbf{z}_e) \quad \nabla C_3(\mathbf{z}_e) \quad \nabla C_4(\mathbf{z}_e) \quad \boldsymbol{\xi}(\mathbf{z}_e) \right]. \tag{6.67}$$

[9] This is the *directional stability* of Hughes [40:121].

Then the projection matrix is given by

$$\mathbf{P}(\mathbf{z}_e) = \mathbf{1} - \mathbf{K}(\mathbf{z}_e)\left(\mathbf{K}^{\mathsf{T}}(\mathbf{z}_e)\mathbf{K}(\mathbf{z}_e)\right)^{-1}\mathbf{K}^{\mathsf{T}}(\mathbf{z}_e). \tag{6.68}$$

From this we may compute the projected Hessian $\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)$ for which we find that the eigenvalues are

$$\left\{0,\ 0,\ 0,\ 0,\ 0,\ 3+k_t,\ 3+k_t,\ \frac{k_t - Q}{3+k_t},\ \frac{4k_t - Q}{2(3+k_t)}\right\}.$$

Four zero eigenvalues are associated with the constraints and one with the cyclic perturbation direction. The remaining four eigenvalues are associated with the restricted tangent space. Thus, the conditions for positive-definiteness on the restricted tangent space are $k_t - Q > 0$ and $4k_t - Q > 0$ which corresponds to the region shown in medium gray in Figure 6.4. Recall that the variational Lagrangian was positive semi-definite in the first quadrant only. This problem clearly demonstrates that the variational Lagrangian need not even be positive semi-definite in order for a relative equilibrium to be a constrained minimum.

6.2.4 Hyperbolic Relative Equilibria.

Spectral Stability. For the hyperbolic relative equilibria, the symmetry axis lies within the plane formed by the tangential vector, $\boldsymbol{\alpha}$, and the orbit normal vector, $\boldsymbol{\beta}$, so that $\gamma_3 = 0$. Furthermore, we have $\beta_3 = Q/k_t$. This condition restricts the permissible range of Q/k_t to $|Q/k_t| \leq 1$ and results in portions of the k_t–μ_4 parameter space being physically unrealizable. For definiteness, we choose the nodal frame such that $\boldsymbol{\gamma}_e$ is in the direction of the first basis vector. The complete set of relative equilibrium conditions for

the hyperbolic case are

$$\Pi_{r_1} = 0 \qquad \Pi_{r_2} = 0 \qquad \Pi_{r_3} = \mu_4 I_a \tag{6.69a}$$

$$\beta_1 = 0 \qquad \beta_2 \text{ unspecified} \qquad \beta_3 = Q/k_t \tag{6.69b}$$

$$\gamma_1 = 1 \qquad \gamma_2 = 0 \qquad \gamma_3 = 0 \tag{6.69c}$$

$$\mu_1 = -I_t \qquad \mu_2 = 3I_t \qquad \mu_3 = 0. \tag{6.69d}$$

By the second Casimir constraint, we must have $\beta_2^2 = 1 - \beta_3^2$. The choice of β_2 as positive or negative corresponds to the positive symmetry axis leaning forward or aft. This makes no difference in the analysis to follow and we leave it unspecified and treat both cases simultaneously. For these values, the linear system matrix becomes

$$\mathbf{A}(\mathbf{z}_e) = \begin{bmatrix} 0 & \frac{Q}{k_t} & -\beta_2 & 0 & 0 & \frac{-k_t\beta_2}{3+k_t} & 0 & 0 & 0 \\ \frac{-Q}{k_t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-3k_t}{3+k_t} \\ (1+k_t)\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{(3+k_t)Q}{k_t} & \frac{(3+k_t)\beta_2}{1+k_t} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{(3+k_t)Q}{k_t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(3+k_t)\beta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3+k_t}{1+k_t} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3+k_t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.70}$$

This matrix does not decouple, but it does block-triangularize so that β_1, β_2, γ_1, and γ_2 are associated with trivial blocks on the diagonal. The characteristic equation again takes the form

$$P(s) = s^5(s^4 + A_2 s^2 + A_0). \tag{6.71}$$

In terms of the inertia parameter, k_t, and the spin parameter, Q, the coefficients of the characteristic equation reduce to

$$A_0 = \frac{3}{k_t}(k_t - Q)(k_t + Q) \qquad A_2 = 1 + 3k_t. \tag{6.72}$$

The spectral stability requirements from Table 3.1 are

$$A_0 \geq 0 \qquad A_2 \geq 0 \qquad A_2^2 - 4A_0 \geq 0 \tag{6.73}$$

which reduce to

$$k_t(k_t^2 - Q^2) \geq 0 \tag{6.74a}$$

$$1 + 3k_t \geq 0 \tag{6.74b}$$

$$k_t(k_t - 6k_t^2 + 9k_t^3 + 12Q^2) \geq 0. \tag{6.74c}$$

However, for physically realizable solutions, we require $|Q/k_t| \leq 1$. Thus Condition (6.74a) reduces to $k_t \geq 0$. Condition (6.74b) is then automatically satisfied. Condition (6.74c) may be written as $k_t\left[(1-3k_t)^2 + 12Q^2\right] \geq 0$ so that it is readily apparent that it too is satisfied for $k_t \geq 0$. Therefore, we need only look at the bounds given by the Condition (6.74a) and the physical constraints. The stability regions are shown in Figure 6.5. We find that only oblate configurations are stable, and then only for very small rotation rates.

Nonlinear Stability. We now apply the projection method to show that the spectrally stable relative equilibria are also nonlinearly stable. Substituting the relative equilibrium conditions into (6.39), we have

$$\nabla^2 F(\mathbf{z}_e) = \operatorname{diag}\left(3 + k_t,\ 3 + k_t,\ \frac{3 + k_t}{1 + k_t},\ 0,\ 0,\ \frac{-k_t}{3 + k_t},\ 0,\ 0,\ \frac{3k_t}{3 + k_t}\right) \tag{6.75}$$

which is clearly indefinite.

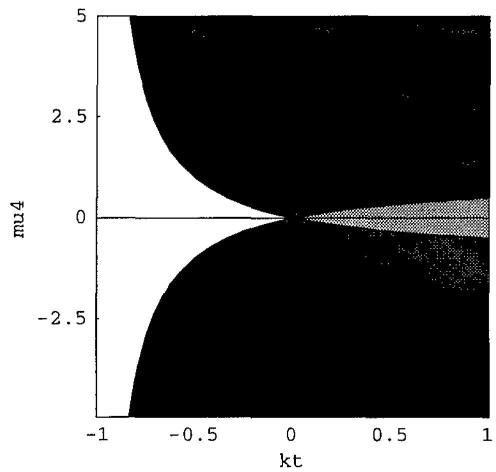


Figure 6.5 Stability Diagram for Hyperbolic Relative Equilibria of Axisymmetric Keplerian System (dark gray = physically impossible, medium gray = nonlinearly and spectrally stable, white = unstable)

We again construct the projection onto the portion of the tangent space orthogonal to the cyclic perturbation direction by forming the matrix $\mathbf{K}(\mathbf{z}_e)$ of basis vectors for $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}(\mathbf{z}_e)\right)$ augmented with the cyclic perturbation vector. We then compute the projected variational Lagrangian $\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)$ for which the eigenvalues are

$$\Bigg\{0,\ 0,\ 0,\ 0,\ 0,\ 3+k_t,\ 3+k_t,\ \frac{3(2k_t^2-Q^2)}{2k_t(3+k_t)},\\ \frac{(3+k_t)(k_t^2-Q^2)}{k_t^2(2k_t^2-Q^2)+2k_t(4k_t^2-Q^2)+(10k_t^2-Q^2)}\Bigg\}.$$

As before, four zero eigenvalues are associated with the constraints and one with the cyclic perturbation direction. The remaining four eigenvalues are associated with the restricted tangent space. The conditions for positive-definiteness on the restricted tangent space are $k_t > 0$ along with the physical constraint $k_t^2 - Q^2 > 0$. Thus, the relative equilibria for oblate configurations are nonlinearly stable.

6.2.5 Conical Relative Equilibria.

Spectral Stability. For the conical relative equilibria, the symmetry axis lies within the plane formed by the orbit normal vector, $\boldsymbol{\beta}$, and the radial vector, $\boldsymbol{\gamma}$, so that $\alpha_3 = 0$. Furthermore, we have $\beta_3 = Q/(4k_t)$. This condition restricts the permissible range of Q/k_t to $|Q/k_t| \leq 4$ and, just as for the hyperbolic case, results in portions of the k_t–μ_4 parameter space being physically unrealizable. We disregard the case where $\gamma_3 = 0$, since this is treated under the cylindrical relative equilibria. For definiteness, we choose the nodal frame such that the second basis vector is in the $\boldsymbol{\alpha}$ direction and the first basis vector is in the plane formed by $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. Then $\gamma_1 = \beta_3$ and $\gamma_3 = -\beta_1$. The complete set of relative equilibrium conditions for the conical case may then be given as

$$\Pi_{r_1} = 0 \qquad \Pi_{r_2} = 0 \qquad \Pi_{r_3} = \mu_4 I_a \tag{6.76a}$$

$$\beta_1 \text{ unspecified} \qquad \beta_2 = 0 \qquad \beta_3 = Q/(4k_t) \tag{6.76b}$$

$$\gamma_1 = Q/(4k_t) \qquad \gamma_2 = 0 \qquad \gamma_3 = -\beta_1 \tag{6.76c}$$

$$\mu_1 = \frac{-16k_t + 3Q^2}{16k_t(3+k_t)} \qquad \mu_2 = \frac{3(16k_t + 16k_t^2 - Q^2)}{16k_t(3+k_t)} \qquad \mu_3 = \frac{3Q\beta_1}{4(3+k_t)} \tag{6.76d}$$

where we have used the Casimir constraint $\beta_1^2 = 1 - \beta_3^3$ in specifying the μ_i. For these values, the linear system matrix becomes

$$\mathbf{A}(\mathbf{z}_e) =$$
$$\begin{bmatrix}
0 & \frac{(1+3k_t)Q}{4k_t} & 0 & 0 & \frac{3Q}{4(3+k_t)} & 0 & 0 & -\frac{3k_t\beta_1}{3+k_t} & 0 \\
-\frac{(1+3k_t)Q}{4k_t} & 0 & \beta_1 & -\frac{3Q}{4(3+k_t)} & 0 & \frac{k_t\beta_1}{3+k_t} & \frac{3k_t\beta_1}{3+k_t} & 0 & -\frac{3Q}{4(3+k_t)} \\
0 & -(1+k_t)\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{(3+k_t)Q}{4k_t} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{(3+k_t)Q}{4k_t} & 0 & -\frac{(3+k_t)\beta_1}{1+k_t} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (3+k_t)\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (3+k_t)\beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-(3+k_t)\beta_1 & 0 & -\frac{(3+k_t)Q}{4k_t(1+k_t)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{(3+k_t)Q}{4k_t} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{6.77}$$

This matrix does not decouple or block-triangularize. However, because we have three independent Casimir functions and a symmetry integral, we are assured the characteristic equation again takes the form

$$P(s) = s^5(s^4 + A_2 s^2 + A_0). \tag{6.78}$$

In terms of the inertia parameter, k_t, and the spin parameter, Q, the coefficients of the characteristic equation are expressed as

$$A_0 = \frac{-3(1-3k_t)(4k_t - Q)(4k_t + Q)}{16k_t} \qquad A_2 = \frac{16k_t - 96k_t^2 + 9Q^2 + 9k_t Q^2}{16k_t}. \tag{6.79}$$

The spectral stability requirements are

$$A_0 \geq 0 \qquad A_2 \geq 0 \qquad A_2^2 - 4A_0 \geq 0 \tag{6.80}$$

which reduce to

$$-k_t(1-3k_t)(4k_t^2 - Q^2) \geq 0 \tag{6.81a}$$

$$-k_t(-16k_t + 96k_t^2 - 9Q^2 - 9k_t Q^2) \geq 0 \tag{6.81b}$$

$$256k_t^2 + 96k_t Q^2 - 864k_t^2 Q^2 - 1728k_t^3 Q^2 + 81Q^4 + 162k_t Q^4 + 81k_t^2 Q^4 \geq 0. \tag{6.81c}$$

However, for physically realizable solutions, we require $|Q/k_t| \leq 4$. Thus, Condition (6.81a) reduces to $-k_t(1-3k_t) \geq 0$. Unlike the hyperbolic case, this does not guarantee either Condition (6.81b) or Condition (6.81c) is satisfied. So we need to look at the bounds given by all three conditions. The stability regions are shown in Figure 6.6. We find that a relatively large region of prolate configurations are spectrally stable while only a very select set of oblate configurations satisfy the conditions. At large rotation rates, relative equilibrium is not physically possible for most configurations.

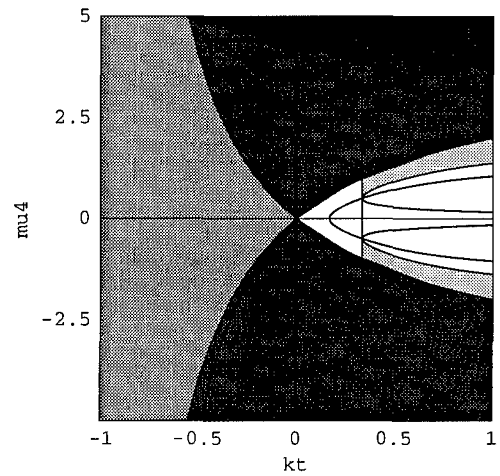


Figure 6.6 Stability Diagram for Conical Relative Equilibria of Axisymmetric Keplerian System (dark gray = physically impossible, medium gray = nonlinearly and spectrally stable, light gray = spectrally stable, white = unstable)

Nonlinear Stability. We again apply the projection method to determine nonlinear stability. We show that the spectrally stable relative equilibria corresponding to prolate configurations are also nonlinearly stable. Substituting the relative equilibrium conditions into (6.39), we have

$$\nabla^2 F(\mathbf{z}_e) = \left[\begin{array}{cccccc} 3+k_t & 0 & 0 & 0 & 0 & 0 \\ 0 & 3+k_t & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3+k_t}{1+k_t} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-3Q^2}{16k_t(3+k_t)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-3Q^2}{16k_t(3+k_t)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-16k_t^2+3Q^2)}{16k_t(3+k_t)} \\ 0 & 0 & 0 & \frac{3Q\beta_1}{4(3+k_t)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3Q\beta_1}{4(3+k_t)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3Q\beta_1}{4(3+k_t)} \end{array}\right. \cdots$$

$$\cdots\left.\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{3Q\beta_1}{4(3+k_t)} & 0 & 0 \\ 0 & \frac{3Q\beta_1}{4(3+k_t)} & 0 \\ 0 & 0 & \frac{3Q\beta_1}{4(3+k_t)} \\ \frac{3(16k_t^2-Q^2)}{16k_t(3+k_t)} & 0 & 0 \\ 0 & \frac{3(16k_t^2-Q^2)}{16k_t(3+k_t)} & 0 \\ 0 & 0 & \frac{3Q^2}{16k_t(3+k_t)} \end{matrix}\right] \tag{6.82}$$

which is not positive definite.

Following the same procedure as in the cylindrical and hyperbolic cases, we construct the projection onto the portion of the tangent space orthogonal to the cyclic perturbation direction by forming the matrix $\mathbf{K}(\mathbf{z}_e)$ of basis vectors for $\mathcal{N}\left(\mathbf{A}^{\mathsf{T}}(\mathbf{z}_e)\right)$ augmented with the cyclic perturbation vector. We then compute the projected variational Lagrangian $\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)$ for which the eigenvalues are

$$\left\{0,\ 0,\ 0,\ 0,\ 0,\ 3+k_t,\ 3+k_t,\ \frac{-3k_t}{3+k_t},\ \frac{(1-3k_t)(3+k_t)(16k_t^2-Q^2)}{48k_t^4+224k_t^3+304k_t^2-Q^2-2k_tQ^2-k_t^2Q^2}\right\}.$$

As before, four zero eigenvalues are associated with the constraints and one with the cyclic perturbation direction. The remaining four eigenvalues are associated with the restricted tangent space. The conditions for positive-definiteness on the restricted tangent space are $k_t < 0$ along with the physical constraint $16k_t^2 - Q^2 > 0$. Thus, the relative equilibria for prolate configurations are nonlinearly stable. This completes the analysis for the Keplerian system.

6.3 Conclusions

We have applied noncanonical Hamiltonian methods to a nondimensional, second-order approximation of the Keplerian system to derive relative equilibrium conditions and eval-

uate the spectral and nonlinear stability of these solutions to the equations of motion. In doing so, we have realized the classical results and validated our assertion that this system is dynamically equivalent to the classical approximation of a rigid body in a central gravitational field. In addition, we considered several different nonlinear stability methods to compare their utility. We found the projection method to be superior due to its simplicity. We will use this method for all subsequent nonlinear stability analyses.

VII. *Free Rigid Body System: Second-Order Approximation*

The previous chapter applied a noncanonical Hamiltonian approach to demonstrate the classical results for the relative equilibria of a rigid body in a central gravitational field. In this chapter and the one to follow, we examine the free rigid body system, which removes the constraint on the trajectory of the center of mass imposed in the Keplerian system and allows for coupling of the orbital and attitude motion. In the current chapter, we continue using the second-order approximation of the potential.

The second-order model of the free rigid-body system has been investigated by Wang *et al* [104] using noncanonical Hamiltonian methods. Their research considered the relative equilibria for an arbitrary rigid body and showed that the principal relative equilibria are still valid solutions. They also showed the nonlinear stability of principal relative equilibria in the Lagrange region "at sufficiently large radius." These same results have previously been found by Beletskii [15] using classical methods.

We demonstrate that the principal relative equilibria are essentially the only solutions (as in the second-order Keplerian system) and determine conditions for spectral and nonlinear stability using the methods applied in the previous chapter for the Keplerian system. We demonstrate nonlinear stability of the Lagrange region for all *practical* configurations.

We also present an analysis of the axisymmetric rigid body, which has not been investigated previously. We demonstrate the existence of three classes of relative equilibria as in the Keplerian system; however, we find that one class — the conical relative equilibria — must follow oblique orbits.

In the following, portions of the development will first be presented without regard to the order of the potential approximation. These sections will be clearly identified as general developments. They will provide points of departure for the second-order model treated here and for the exact model to be examined in Chapter 8.

7.1 Relative Equilibria for an Arbitrary Rigid Body

7.1.1 Nondimensional Hamiltonian System (General Development). The nondimensionalization of the Hamiltonian system as presented here and the reduction of the relative equilibrium conditions to be presented in the next section follow the analysis of Wang *et al* [105]. We begin with the free rigid-body Hamiltonian system given in Chapter 5 as

$$\begin{pmatrix} \dot{\bar{\mathbf{\Sigma}}} \\ \dot{\bar{\mathbf{\Lambda}}} \\ \dot{\bar{\mathbf{\Pi}}} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{1} & \bar{\mathbf{\Sigma}}^{\times} \\ \mathbf{1} & \mathbf{0} & \bar{\mathbf{\Lambda}}^{\times} \\ \bar{\mathbf{\Sigma}}^{\times} & \bar{\mathbf{\Lambda}}^{\times} & \bar{\mathbf{\Pi}}^{\times} \end{bmatrix} \begin{pmatrix} \frac{1}{\bar{m}}\bar{\mathbf{\Sigma}} \\ \nabla \bar{V}(\bar{\mathbf{\Lambda}}) \\ \bar{\mathbf{I}}^{-1}\bar{\mathbf{\Pi}} \end{pmatrix} \tag{7.1}$$

where $\bar{\mathbf{\Sigma}}$, $\bar{\mathbf{\Lambda}}$, and $\bar{\mathbf{\Pi}}$ are the linear momentum, position, and angular momentum about the center of mass, respectively. Again, the overbar denotes dimensional variables. To nondimensionalize the system, we use the mass, length, and time scales

$$m = \bar{m} = \int_{\mathcal{B}} d\bar{m} \qquad l = \left(\frac{\operatorname{tr}(\bar{\mathbf{I}})}{\bar{m}}\right)^{\frac{1}{2}} \qquad t = \left(\frac{l^3}{G_*}\right)^{\frac{1}{2}}. \tag{7.2}$$

Note that the time scale which was used for the Keplerian system is not appropriate here since the orbital frequency is no longer a constant. The resulting nondimensional system is

$$\begin{pmatrix} \dot{\mathbf{\Sigma}} \\ \dot{\mathbf{\Lambda}} \\ \dot{\mathbf{\Pi}} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{1} & \mathbf{\Sigma}^{\times} \\ \mathbf{1} & \mathbf{0} & \mathbf{\Lambda}^{\times} \\ \mathbf{\Sigma}^{\times} & \mathbf{\Lambda}^{\times} & \mathbf{\Pi}^{\times} \end{bmatrix} \begin{pmatrix} \mathbf{\Sigma} \\ \nabla V(\mathbf{\Lambda}) \\ \mathbf{I}^{-1}\mathbf{\Pi} \end{pmatrix} \tag{7.3}$$

where $\mathbf{\Sigma} = (m^{-1}l^{-1}t)\bar{\mathbf{\Sigma}}$, $\mathbf{\Lambda} = l^{-1}\bar{\mathbf{\Lambda}}$, $\mathbf{\Pi} = (m^{-1}l^{-2}t)\bar{\mathbf{\Pi}}$, and $\mathbf{I} = (m^{-1}l^{-2})\bar{\mathbf{I}}$. The Hamiltonian for this system is

$$H(\mathbf{z}) = \frac{1}{2}\mathbf{\Pi}\cdot\mathbf{I}^{-1}\mathbf{\Pi} + \frac{1}{2}\mathbf{\Sigma}\cdot\mathbf{\Sigma} + V(\mathbf{\Lambda}). \tag{7.4}$$

The nondimensional form of the Casimir function for this system is

$$C_1(\mathbf{z}) = \frac{1}{2}\left|\mathbf{\Pi} + \mathbf{\Lambda}^{\times}\mathbf{\Sigma}\right|^2. \tag{7.5}$$

This and the Hamiltonian are the two first integrals for this system.

7.1.2 Relative Equilibrium Conditions and Linearization (General Development). To determine the relative equilibria for this system, we consider the critical points of the variational Lagrangian

$$F(\mathbf{z}) = H(\mathbf{z}) - \mu_1 C_1(\mathbf{z}). \tag{7.6}$$

The resulting relative equilibrium conditions are

$$\boldsymbol{\Sigma}_e + \mu_1 \boldsymbol{\Lambda}_e^{\times} \left(\boldsymbol{\Pi}_e + \boldsymbol{\Lambda}_e^{\times} \boldsymbol{\Sigma}_e\right) = \mathbf{0} \tag{7.7a}$$

$$\nabla V(\boldsymbol{\Lambda}_e) - \mu_1 \boldsymbol{\Sigma}_e^{\times} \left(\boldsymbol{\Pi}_e + \boldsymbol{\Lambda}_e^{\times} \boldsymbol{\Sigma}_e\right) = \mathbf{0} \tag{7.7b}$$

$$\mathbf{I}^{-1} \boldsymbol{\Pi}_e - \mu_1 \left(\boldsymbol{\Pi}_e + \boldsymbol{\Lambda}_e^{\times} \boldsymbol{\Sigma}_e\right) = \mathbf{0} \tag{7.7c}$$

along with the Casimir constraint given by Equation (7.5). For convenience, let $\boldsymbol{\Gamma} = \boldsymbol{\Pi} + \boldsymbol{\Lambda}^{\times} \boldsymbol{\Sigma}$ be the angular momentum about the center of attraction. In terms of the angular velocity $\boldsymbol{\Omega}_e = \mathbf{I}^{-1} \boldsymbol{\Pi}_e$, the relative equilibrium conditions become

$$\boldsymbol{\Sigma}_e + \mu_1 \boldsymbol{\Lambda}_e^{\times} \boldsymbol{\Gamma}_e = \mathbf{0} \tag{7.8a}$$

$$\nabla V(\boldsymbol{\Lambda}_e) - \mu_1 \boldsymbol{\Sigma}_e^{\times} \boldsymbol{\Gamma}_e = \mathbf{0} \tag{7.8b}$$

$$\boldsymbol{\Omega}_e - \mu_1 \boldsymbol{\Gamma}_e = \mathbf{0}. \tag{7.8c}$$

Solving Equation (7.8c) for $\mu_1 \boldsymbol{\Gamma}_e$, the first two conditions may be written as

$$\boldsymbol{\Sigma}_e + \boldsymbol{\Lambda}_e^{\times} \boldsymbol{\Omega}_e = \mathbf{0} \tag{7.9a}$$

$$\nabla V(\boldsymbol{\Lambda}_e) - \boldsymbol{\Sigma}_e^{\times} \boldsymbol{\Omega}_e = \mathbf{0}. \tag{7.9b}$$

Equation (7.9a) reduces to $\boldsymbol{\Sigma}_e = -\boldsymbol{\Lambda}_e^{\times} \boldsymbol{\Omega}_e = \boldsymbol{\Omega}_e^{\times} \boldsymbol{\Lambda}_e$. This enables us to eliminate $\boldsymbol{\Sigma}_e$ from Equations (7.9b) and (7.8c). Then the relative equilibrium conditions may be expressed

in terms of $\boldsymbol{\Omega}_e$, $\boldsymbol{\Lambda}_e$, and μ_1 as

$$\nabla V(\boldsymbol{\Lambda}_e) + \boldsymbol{\Omega}_e{}^\times \boldsymbol{\Omega}_e{}^\times \boldsymbol{\Lambda}_e = \mathbf{0} \tag{7.10a}$$

$$\boldsymbol{\Omega}_e - \mu_1 \mathbf{I} \boldsymbol{\Omega}_e + \mu_1 \boldsymbol{\Lambda}_e{}^\times \boldsymbol{\Lambda}_e{}^\times \boldsymbol{\Omega}_e = \mathbf{0}. \tag{7.10b}$$

Furthermore, in light of Equation (7.8c), the Casimir condition given in Equation (7.5) becomes

$$C_1 = \frac{1}{2} \left| \frac{\boldsymbol{\Omega}_e}{\mu_1} \right|^2 \tag{7.11}$$

where C_1 without the argument is a constant representing the value of the first integral. Wang *et al* [105] introduce $\beta = 1/\mu_1$ and $c = \mu_1^2 C_1$ so that Equations (7.10) and (7.11) may be written as

$$\nabla V(\boldsymbol{\Lambda}_e) + \boldsymbol{\Omega}_e{}^\times \boldsymbol{\Omega}_e{}^\times \boldsymbol{\Lambda}_e = \mathbf{0} \tag{7.12a}$$

$$\beta \boldsymbol{\Omega}_e - \mathbf{I} \boldsymbol{\Omega}_e + \boldsymbol{\Lambda}_e{}^\times \boldsymbol{\Lambda}_e{}^\times \boldsymbol{\Omega}_e = \mathbf{0} \tag{7.12b}$$

$$c - \frac{1}{2} \boldsymbol{\Omega}_e \cdot \boldsymbol{\Omega}_e = 0. \tag{7.12c}$$

Wang *et al* [105] point out that this is equivalent to the variational problem

$$\begin{gathered} \text{Make stationary } L(\boldsymbol{\Lambda}, \boldsymbol{\Omega}) = \frac{1}{2} \boldsymbol{\Omega} \cdot \mathbf{I} \boldsymbol{\Omega} + \frac{1}{2} \left| \boldsymbol{\Omega}^\times \boldsymbol{\Lambda} \right| - V(\boldsymbol{\Lambda}) \\ \text{subject to } c - \frac{1}{2} \boldsymbol{\Omega} \cdot \boldsymbol{\Omega} = 0. \end{gathered} \tag{7.13}$$

Note that this is different from the variational characterization discussed in Chapter 3 related to the variational Lagrangian $F(\mathbf{z}) = H(\mathbf{z}) - \mu_1 C_1(\mathbf{z})$. The function for which this problem seeks constrained extrema is the Lagrangian (of mechanics) rather than the Hamiltonian. Wang *et al* [105] use this variational problem as a starting point to prove the existence of oblique relative equilibria for asymmetric rigid bodies. We will return to this characterization of the relative equilibria in Chapter 8.

Equations (7.12) form a nonlinear system of seven equations in seven unknowns ($\boldsymbol{\Omega}_e$, $\boldsymbol{\Lambda}_e$, and β) parameterized by c. This is the general form of the relative equilibrium conditions which we will use to determine the relative equilibria for both the second-order model treated in this chapter and the exact model considered in Chapter 8. However, for linearization of the equations of motion and stability analysis we will return to the form given in Equation (7.7) upon which our methods are based.

Recall from Chapter 3 that the linearization of Equation (7.3) takes the form

$$\delta\dot{\mathbf{z}} = \mathbf{A}(\mathbf{z}_e)\,\delta\mathbf{z}. \tag{7.14}$$

where $\mathbf{z} = \mathbf{z}_e + \delta\mathbf{z}$ and the linear system matrix $\mathbf{A}(\mathbf{z}) = \mathbf{J}(\mathbf{z})\,\nabla^2 F(\mathbf{z})$. The second variation of $F(\mathbf{z})$ is

$$\nabla^2 F(\mathbf{z}) = \begin{bmatrix} \mathbf{1} + \mu_1 \boldsymbol{\Lambda}^\times \boldsymbol{\Lambda}^\times & -\mu_1 \boldsymbol{\Gamma}^\times - \mu_1 \boldsymbol{\Lambda}^\times \boldsymbol{\Sigma}^\times & \mu_1 \boldsymbol{\Lambda}^\times \\ \mu_1 \boldsymbol{\Gamma}^\times - \mu_1 \boldsymbol{\Sigma}^\times \boldsymbol{\Lambda}^\times & \nabla^2 V(\boldsymbol{\Lambda}) + \mu_1 \boldsymbol{\Sigma}^\times \boldsymbol{\Sigma}^\times & -\mu_1 \boldsymbol{\Sigma}^\times \\ -\mu_1 \boldsymbol{\Lambda}^\times & \mu_1 \boldsymbol{\Sigma}^\times & \mathbf{I}^{-1} - \mu_1 \mathbf{1} \end{bmatrix} \tag{7.15}$$

where again for convenience we introduce the vector $\boldsymbol{\Gamma} = \boldsymbol{\Pi} + \boldsymbol{\Lambda}^\times \boldsymbol{\Sigma}$ to represent the angular momentum about the center of attraction. Premultiplying this matrix by $\mathbf{J}(\mathbf{z})$ and simplifying, we find that linearization about a relative equilibrium gives

$$\mathbf{A}(\mathbf{z}_e) = \begin{bmatrix} -\boldsymbol{\Omega}_e{}^\times & -\nabla^2 V(\boldsymbol{\Lambda}_e) & \boldsymbol{\Sigma}_e{}^\times \mathbf{I}^{-1} \\ \mathbf{1} & -\boldsymbol{\Omega}_e{}^\times & \boldsymbol{\Lambda}_e{}^\times \mathbf{I}^{-1} \\ \mathbf{0} & \boldsymbol{\Lambda}_e{}^\times \nabla^2 V(\boldsymbol{\Lambda}_e) + \left(\boldsymbol{\Omega}_e{}^\times \boldsymbol{\Sigma}_e\right)^\times & (\mathbf{I}\boldsymbol{\Omega}_e)^\times \mathbf{I}^{-1} - \boldsymbol{\Omega}_e{}^\times \end{bmatrix}. \tag{7.16}$$

In deriving this matrix we have made use of Equation (7.8c).

Before focusing on the second-order model, we derive some general results from the relative equilibrium conditions. We begin by taking the dot and cross products of $\boldsymbol{\Lambda}_e$ and $\boldsymbol{\Omega}_e$ with Equation (7.12a) to get

$$\boldsymbol{\Lambda}_e \cdot \nabla V(\boldsymbol{\Lambda}_e) = |\boldsymbol{\Omega}_e|^2\,|\boldsymbol{\Lambda}_e|^2 - (\boldsymbol{\Omega}_e \cdot \boldsymbol{\Lambda}_e)^2 \tag{7.17a}$$

$$\mathbf{\Lambda}_e{}^{\times}\nabla V(\mathbf{\Lambda}_e) = (\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e)\mathbf{\Omega}_e{}^{\times}\mathbf{\Lambda}_e \tag{7.17b}$$

$$\mathbf{\Omega}_e \cdot \nabla V(\mathbf{\Lambda}_e) = 0 \tag{7.17c}$$

$$\mathbf{\Omega}_e{}^{\times}\nabla V(\mathbf{\Lambda}_e) = |\mathbf{\Omega}_e|^2\,\mathbf{\Omega}_e{}^{\times}\mathbf{\Lambda}_e. \tag{7.17d}$$

Likewise, the dot and cross products of $\mathbf{\Lambda}_e$ and $\mathbf{\Omega}_e$ with Equation (7.12b) reduce to

$$\mathbf{\Lambda}_e \cdot \mathbf{I}\mathbf{\Omega}_e = \beta(\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e) \tag{7.18a}$$

$$\mathbf{\Lambda}_e{}^{\times}\mathbf{I}\mathbf{\Omega}_e = (\beta - |\mathbf{\Lambda}_e|^2)\mathbf{\Lambda}_e{}^{\times}\mathbf{\Omega}_e \tag{7.18b}$$

$$\mathbf{\Omega}_e \cdot \mathbf{I}\mathbf{\Omega}_e = (\beta - |\mathbf{\Lambda}_e|^2)\,|\mathbf{\Omega}_e|^2 + (\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e)^2 \tag{7.18c}$$

$$\mathbf{\Omega}_e{}^{\times}\mathbf{I}\mathbf{\Omega}_e = (\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e)\mathbf{\Omega}_e{}^{\times}\mathbf{\Lambda}_e. \tag{7.18d}$$

These equations, along with the original relative equilibrium conditions, allow us to make some general remarks regarding the nature of relative equilibria for the free rigid-body system:

Remark 1. Solving Equation (7.18c) for β gives

$$\beta = \frac{1}{|\mathbf{\Omega}_e|^2}\left[\mathbf{\Omega}_e \cdot \mathbf{I}\mathbf{\Omega}_e + |\mathbf{\Lambda}_e|^2\,|\mathbf{\Omega}_e|^2 - (\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e)^2\right]. \tag{7.19}$$

Positive-definiteness of $\mathbf{I}$ implies $\mathbf{\Omega}_e \cdot \mathbf{I}\mathbf{\Omega}_e > 0$ for $\mathbf{\Omega}_e \neq \mathbf{0}$ while the Cauchy-Schwartz inequality gives $|\mathbf{\Lambda}_e|\,|\mathbf{\Omega}_e| \geq |\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e|$. Thus, β is guaranteed to be positive (as is the Lagrange multiplier μ_1).

Remark 2. By consideration of Equations (7.9a) and (7.9b), we find that the vectors $\mathbf{\Sigma}_e$, $\mathbf{\Omega}_e$, and $\nabla V(\mathbf{\Lambda}_e)$ form an orthogonal triplet, similar to the vectors $\boldsymbol{\alpha}_e$, $\boldsymbol{\beta}_e$, and $\boldsymbol{\gamma}_e$ of the Keplerian system. In addition, by combining Equations (7.17b) and (7.18d) we find

$$\mathbf{\Lambda}_e{}^{\times}\nabla V(\mathbf{\Lambda}_e) - \mathbf{\Omega}_e{}^{\times}\mathbf{I}\mathbf{\Omega}_e = \mathbf{0}. \tag{7.20}$$

This is the form of the Likins-Roberson Condition for the free rigid-body system. It can only be satisfied if $\mathbf{\Lambda}_e$ and $\mathbf{\Pi}_e = \mathbf{I}\mathbf{\Omega}_e$ are in the plane formed by $\mathbf{\Omega}_e$ and $\nabla V(\mathbf{\Lambda}_e)$. Positive definiteness of $\mathbf{I}$ assures that $\mathbf{\Omega}_e \cdot \mathbf{I}\mathbf{\Omega}_e$ is positive, while the Cauchy-Schwarz inequality applied to Equation (7.17a) assures that $\mathbf{\Lambda}_e \cdot \nabla V(\mathbf{\Lambda}_e)$ is positive. Finally, by Equation (7.18a) and Remark 1, we find that $\mathbf{\Lambda}_e \cdot \mathbf{I}\mathbf{\Omega}_e$ has the same sign as $\mathbf{\Lambda}_e \cdot \mathbf{\Omega}_e$. From this we may conclude that the relative equilibria assume one of the geometric configurations shown in Figure 7.1. In these diagrams, we let the angular velocity define a polar axis and refer to oblique orbits with positive and negative values of $\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e$ as being in the northern and southern hemisphere, respectively.

Remark 3. If we assume the orbit is orthogonal, then Equations (7.17) and (7.18) reduce to

$$\mathbf{\Lambda}_e \cdot \nabla V(\mathbf{\Lambda}_e) = |\mathbf{\Omega}_e|^2 \, |\mathbf{\Lambda}_e|^2 \tag{7.21a}$$

$$\mathbf{\Lambda}_e{}^{\times} \nabla V(\mathbf{\Lambda}_e) = \mathbf{0} \tag{7.21b}$$

$$\mathbf{\Omega}_e \cdot \nabla V(\mathbf{\Lambda}_e) = 0 \tag{7.21c}$$

$$\mathbf{\Omega}_e{}^{\times} \nabla V(\mathbf{\Lambda}_e) = |\mathbf{\Omega}_e|^2 \, \mathbf{\Omega}_e{}^{\times} \mathbf{\Lambda}_e \tag{7.21d}$$

and

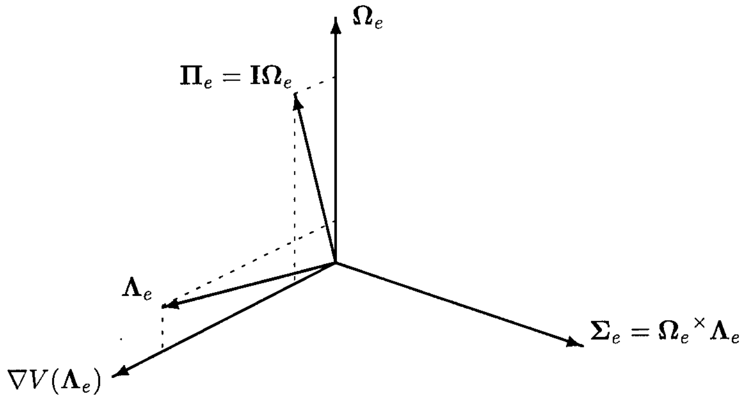
$$\mathbf{\Lambda}_e \cdot \mathbf{I}\mathbf{\Omega}_e = 0 \tag{7.22a}$$

$$\mathbf{\Lambda}_e{}^{\times} \mathbf{I}\mathbf{\Omega}_e = (\beta - |\mathbf{\Lambda}_e|^2) \mathbf{\Lambda}_e{}^{\times} \mathbf{\Omega}_e \tag{7.22b}$$

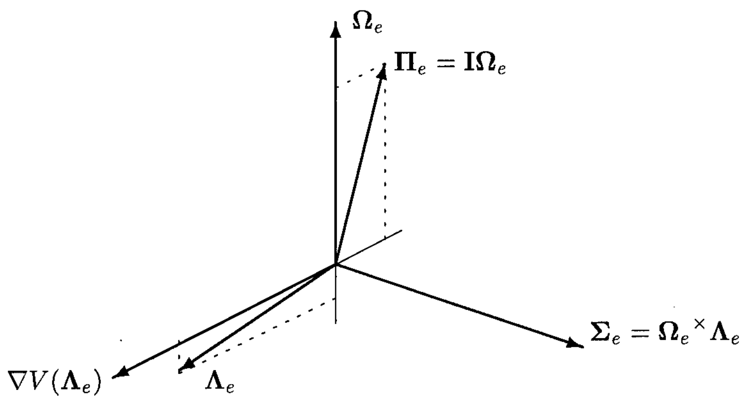
$$\mathbf{\Omega}_e \cdot \mathbf{I}\mathbf{\Omega}_e = (\beta - |\mathbf{\Lambda}_e|^2) \, |\mathbf{\Omega}_e|^2 \tag{7.22c}$$

$$\mathbf{\Omega}_e{}^{\times} \mathbf{I}\mathbf{\Omega}_e = \mathbf{0}. \tag{7.22d}$$

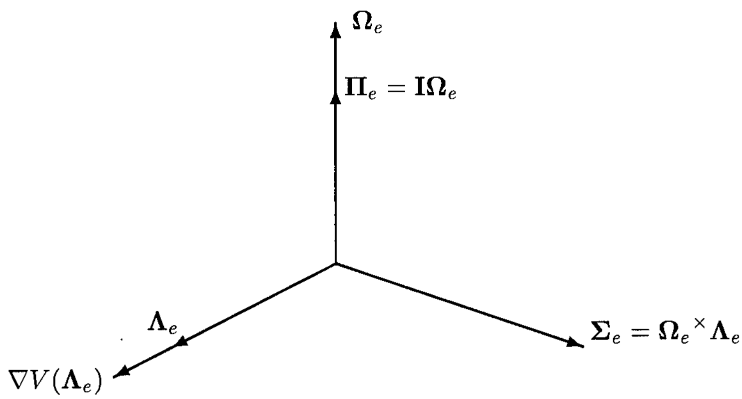
We find that the force is aligned with the radial direction ($\nabla V(\mathbf{\Lambda}_e)$ and $\mathbf{\Lambda}_e$ are parallel), the angular momentum is aligned with the angular velocity ($\mathbf{I}\mathbf{\Omega}_e$ and $\mathbf{\Omega}_e$ are parallel) so that the rotation is about a principal axis, and the gravitational torque is identically zero ($\mathbf{\Lambda}_e{}^{\times} \nabla V(\mathbf{\Lambda}_e)$ is zero). The argument is reversible for each



(a) Oblique Orbit (Northern Hemisphere)



(b) Oblique Orbit (Southern Hemisphere)



(c) Orthogonal Orbit

Figure 7.1 Relative Equilibrium Geometry for the Free Rigid-Body System

of these properties, so that they are unique to orthogonal relative equilibria. It is the converse that is then the surprising result: for oblique relative equilibria, the force is *not* in the radial direction, the body rotates about an axis which is *not* a principal axis, and a *nonzero* gravitational torque acts on the body about the tangential axis.

Remark 4. Equation (7.21a) may be solved for $|\mathbf{\Omega}_e|$ to give a generalized form of the Kepler frequency for orthogonal orbits:

$$|\mathbf{\Omega}_e| = \frac{1}{|\mathbf{\Lambda}_e|} \left[\mathbf{\Lambda}_e \cdot \nabla V(\mathbf{\Lambda}_e)\right]^{1/2}. \tag{7.23}$$

Since we have shown that for orthogonal orbits the force is aligned with the radial, this reduces to

$$|\mathbf{\Omega}_e| = \left(\frac{|\nabla V(\mathbf{\Lambda}_e)|}{|\mathbf{\Lambda}_e|}\right)^{1/2} \tag{7.24}$$

which has been derived previously by Beletskii [15]. For a particle or spherically symmetric body, the gradient of the potential is $\mathbf{\Lambda}_e / |\mathbf{\Lambda}_e|^3$ and this reduces to the nondimensional form of the standard Kepler frequency $|\mathbf{\Omega}_e| = |\mathbf{\Lambda}_e|^{-3/2}$. If we apply the same method to Equation (7.17a), we obtain

$$|\mathbf{\Omega}_e| = \frac{1}{|\mathbf{\Lambda}_e|} \left[\mathbf{\Lambda}_e \cdot \nabla V(\mathbf{\Lambda}_e) + (\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e)^2\right]^{1/2} \tag{7.25}$$

which is a generalization of the Kepler frequency valid for all relative equilibria.

We will return to these remarks below and again in Chapter 8.

7.1.3 Uniqueness of the Principal Relative Equilibria (2nd-Order Approximation). We now turn to the specifics of the second-order potential approximation and examine the relative equilibrium conditions of Equation (7.12) applied to this model. Wang *et al* [104] have shown that the relative equilibria must be orthogonal for sufficiently large $|\mathbf{\Lambda}_e|$. Use of a nondimensional formulation enables us to refine this result by reducing the minimum radial distance for which it applies to $|\mathbf{\Lambda}_e| = 3/2$. We then show that all relative

equilibria satisfying this condition are not only orthogonal, but are also principal.[1] That is, the principal axes are aligned with the orbital frame. Thus, the classical result of Likins and Roberson [55] is extended to the case of coupled orbital and attitude motion with a second-order potential approximation.

The second-order approximation of the potential, as a function of $\mathbf{\Lambda}$, is given in Appendix B as

$$\bar{V}_2(\bar{\mathbf{\Lambda}}) = -\frac{\bar{G}_*\bar{m}}{|\bar{\mathbf{\Lambda}}|} + \frac{\bar{G}_*\bar{\mathbf{\Lambda}} \cdot \bar{\chi}}{|\bar{\mathbf{\Lambda}}|^3} - \frac{1}{2}\frac{\bar{G}_* \operatorname{tr}\left(\bar{\mathbf{I}}\right)}{|\bar{\mathbf{\Lambda}}|^3} + \frac{3}{2}\frac{\bar{G}_*\bar{\mathbf{\Lambda}} \cdot \bar{\mathbf{I}}\bar{\mathbf{\Lambda}}}{|\bar{\mathbf{\Lambda}}|^5}. \tag{B.33}$$

We again assume the origin of the body frame is located at the center of mass so that the second term on the right-hand side is eliminated. This potential takes the nondimensional form

$$V_2(\mathbf{\Lambda}) = -\frac{1}{|\mathbf{\Lambda}|} - \frac{1}{2}\frac{1}{|\mathbf{\Lambda}|^3} + \frac{3}{2}\frac{\mathbf{\Lambda} \cdot \mathbf{I}\mathbf{\Lambda}}{|\mathbf{\Lambda}|^5}. \tag{7.26}$$

Taking the gradient gives

$$\nabla V_2(\mathbf{\Lambda}) = \left(\frac{1}{|\mathbf{\Lambda}|^3} + \frac{3}{2}\frac{1}{|\mathbf{\Lambda}|^5} - \frac{15}{2}\frac{\mathbf{\Lambda} \cdot \mathbf{I}\mathbf{\Lambda}}{|\mathbf{\Lambda}|^7}\right)\mathbf{\Lambda} + \frac{3}{|\mathbf{\Lambda}|^5}\mathbf{I}\mathbf{\Lambda}. \tag{7.27}$$

The force $\mathbf{F} = -\nabla V_2(\mathbf{\Lambda})$ is not directed along the radial, in general, due to the last term which is not a scalar multiple of $\mathbf{\Lambda}$ unless the body has three equal moments of inertia. This suggests that general solutions of the equations of motion are not confined to orbits which lie in an invariant plane containing the center of attraction, as is the case for the particle moving in a central gravitational field.[2] In spite of this fact, we will now show that the second-order model does not allow oblique relative equilibria except possibly at extremely small radii.

[1] In Ref. [104], the authors claim to show that the principal relative equilibria are the unique solution. However, the proof presented is only sufficient to show orthogonality. The nonexistence of nonprincipal orthogonal relative equilibria cannot be assumed as we show in the next chapter.

[2] In fact, general solutions are not restricted to any plane. It is the definition of relative equilibrium that restricts these solutions to a plane.

In the previous section, we showed that at relative equilibrium the force must lie in a fixed plane for which the angular velocity is a normal vector. Inserting the second-order potential into Equation (7.17c), we get

$$
\begin{aligned}
0 &= \mathbf{\Omega}_e \cdot \nabla V_2(\mathbf{\Lambda}_e) \\
&= \left(\frac{1}{|\mathbf{\Lambda}_e|^3} + \frac{3}{2}\frac{1}{|\mathbf{\Lambda}_e|^5} - \frac{15}{2}\frac{\mathbf{\Lambda}_e \cdot \mathbf{I}\mathbf{\Lambda}_e}{|\mathbf{\Lambda}_e|^7} \right) (\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e) + \frac{3}{|\mathbf{\Lambda}_e|^5}(\mathbf{\Omega}_e \cdot \mathbf{I}\mathbf{\Lambda}_e).
\end{aligned}
\tag{7.28}
$$

By Equation (7.18a) and the symmetry of $\mathbf{I}$ this reduces to

$$
0 = \frac{1}{|\mathbf{\Lambda}_e|^3} \left(1 + \frac{3}{2}\frac{(1+2\beta)}{|\mathbf{\Lambda}_e|^2} - \frac{15}{2}\frac{\mathbf{\Lambda}_e \cdot \mathbf{I}\mathbf{\Lambda}_e}{|\mathbf{\Lambda}_e|^4} \right) (\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e). \tag{7.29}
$$

But for $|\mathbf{\Lambda}_e| \geq 3/2$, we have

$$
\begin{aligned}
&(1 + \frac{3}{2}\frac{(1+2\beta)}{|\mathbf{\Lambda}_e|^2} - \frac{15}{2}\frac{\mathbf{\Lambda}_e \cdot \mathbf{I}\mathbf{\Lambda}_e}{|\mathbf{\Lambda}_e|^4} \Big) \\
&> \left(1 + \frac{3}{2}\frac{1}{|\mathbf{\Lambda}_e|^2} - \frac{15}{2}\frac{\mathbf{\Lambda}_e \cdot \mathbf{I}\mathbf{\Lambda}_e}{|\mathbf{\Lambda}_e|^4} \right) && (\text{since } \beta > 0) \\
&> \left(1 + \frac{3}{2}\frac{1}{|\mathbf{\Lambda}_e|^2} - \frac{15}{4}\frac{1}{|\mathbf{\Lambda}_e|^2} \right) && (\text{since } \mathbf{\Lambda}_e \cdot \mathbf{I}\mathbf{\Lambda}_e \leq I_{\max} |\mathbf{\Lambda}_e|^2 < \frac{1}{2}|\mathbf{\Lambda}_e|^2) \\
&= \left(1 - \frac{9}{4}\frac{1}{|\mathbf{\Lambda}_e|^2} \right) \\
&\geq 0 && (\text{since } |\mathbf{\Lambda}_e| \geq 3/2)
\end{aligned}
$$

so that the previous equation can only be satisfied when $\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e = 0$. Hence, the relative equilibria are all orthogonal.

Returning to the relative equilibrium condition given in Equation (7.12a), the second-order potential gives

$$
\left(\frac{1}{|\mathbf{\Lambda}|^3} + \frac{3}{2}\frac{1}{|\mathbf{\Lambda}|^5} - \frac{15}{2}\frac{\mathbf{\Lambda} \cdot \mathbf{I}\mathbf{\Lambda}}{|\mathbf{\Lambda}|^7} \right) \mathbf{\Lambda} + \frac{3}{|\mathbf{\Lambda}|^5}\mathbf{I}\mathbf{\Lambda} = -\mathbf{\Omega}_e{}^{\times}\mathbf{\Omega}_e{}^{\times}\mathbf{\Lambda}_e. \tag{7.30}
$$

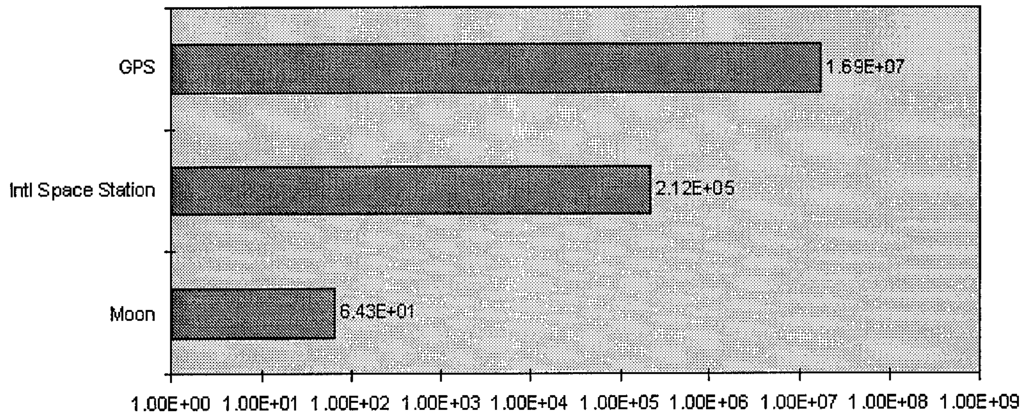


Figure 7.2 Value of Nondimensional Radius for Typical Satellites

Since the relative equilibria are orthogonal, this can be expressed as

$$\mathbf{I}\mathbf{\Lambda}_e = \frac{|\mathbf{\Lambda}_e|^5}{3}\left(|\mathbf{\Omega}_e|^2 - \frac{1}{|\mathbf{\Lambda}|^3} - \frac{3}{2}\frac{1}{|\mathbf{\Lambda}|^5} + \frac{15}{2}\frac{\mathbf{\Lambda}\cdot\mathbf{I}\mathbf{\Lambda}}{|\mathbf{\Lambda}|^7}\right)\mathbf{\Lambda} \tag{7.31}$$

which indicates the radial vector is along a principal axis. By Remark 3 of the previous section, for orthogonal orbits the angular velocity is also along a principal axis. Thus, the relative equilibria must be principal relative equilibria. This result is valid for all relative equilibria satisfying the requirement $|\mathbf{\Lambda}_e| \geq 3/2$. Figure 7.2 shows several typical values of $|\mathbf{\Lambda}_e|$ and illustrates that for all practical applications the requirement is met.

While not of great practical utility, it is interesting from a dynamical systems viewpoint to note that for some of the principal relative equilibria the radius is bounded from below. By Equation (7.17a) and the Cauchy-Schwarz inequality, we have $\mathbf{\Lambda}_e \cdot \nabla V(\mathbf{\Lambda}_e) \geq 0$. Inserting the second-order potential approximation of Equation (7.26), we find

$$\frac{1}{|\mathbf{\Lambda}|} + \frac{3}{2}\frac{1}{|\mathbf{\Lambda}|^3} - \frac{9}{2}\frac{\mathbf{\Lambda}\cdot\mathbf{I}\mathbf{\Lambda}}{|\mathbf{\Lambda}|^5} \geq 0 \tag{7.32}$$

which reduces to

$$|\mathbf{\Lambda}_e|^2 \geq \frac{9}{2}I_\Lambda - \frac{3}{2} \tag{7.33}$$

where $I_\Lambda = \mathbf{\Lambda}_e \cdot \mathbf{I}\mathbf{\Lambda}_e / |\mathbf{\Lambda}_e|^2$ is bounded by the minimum and maximum inertias. For a spherically symmetric body, the nondimensionalization of the problem requires $I_\Lambda = 1/3$ for which Equation (7.33) gives $|\mathbf{\Lambda}_e| \geq 0$. However, for bodies with at least two unequal inertias, one of these inertias must be larger than one third (but less than one half). For the principal relative equilibria which have this principal axis along the radial direction, the radius is bounded by

$$\Lambda_0 = \left(\frac{9}{2}I_\Lambda - \frac{3}{2}\right)^{\frac{1}{2}} \tag{7.34}$$

where $0 < \Lambda_0 < \sqrt{3/4}$. The existence of a lower bound is suggestive that there is a radius at which these principal relative equilibria bifurcate into oblique relative equilibria as the distance of the body from the axis of rotation goes to zero. Investigation of the dynamics at small radius is suggested as a potential topic for future research.

7.1.4 Spectral Stability (2nd-Order Approximation). Without loss of generality, we consider the principal relative equilibrium for which the body frame is aligned with the orbital frame. Let $R = |\mathbf{\Lambda}_e|$ and $\omega = |\mathbf{\Omega}_e|$ so that the relative equilibrium is given by

$$\mathbf{\Lambda}_e = (0,0,R) \qquad \mathbf{\Omega}_e = (0,\omega,0) \qquad \mathbf{\Sigma}_e = (\omega R, 0, 0) \qquad \beta = I_2 + R^2 \tag{7.35}$$

where the last equality comes from Equation (7.19). Furthermore, the generalized Keplerian frequency condition of Equation (7.24) gives the relationship between R and ω as

$$\omega^2 = \frac{1}{R^3}\left[1 + \frac{3}{2R^2}(1 - 3I_3)\right]. \tag{7.36}$$

The general form of the linear system matrix is given in Equation (7.16). Inserting the relative equilibrium conditions of Equation (7.35), this becomes

$$\mathbf{A}(\mathbf{z}_e) =$$

$$\begin{bmatrix} 0 & 0 & -\omega & -v_{11} & -v_{12} & -v_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & -v_{12} & -v_{22} & -v_{23} & 0 & 0 & \frac{-\omega R}{I_3} \\ \omega & 0 & 0 & -v_{13} & -v_{23} & -v_{33} & 0 & \frac{\omega R}{I_2} & 0 \\ 1 & 0 & 0 & 0 & 0 & -\omega & 0 & \frac{-R}{I_2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{R}{I_1} & 0 & 0 \\ 0 & 0 & 1 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -v_{12}R & -(v_{22}-\omega^2)R & -v_{23}R & 0 & 0 & \frac{-\omega(I_3-I_2)}{I_3} \\ 0 & 0 & 0 & (v_{11}-\omega^2)R & v_{12}R & v_{13}R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\omega(I_1-I_2)}{I_1} & 0 & 0 \end{bmatrix} \tag{7.37}$$

where v_{ij} is the ijth element of $\nabla^2 V_2(\mathbf{\Lambda}_e)$. For the second-order model, the Hessian of the potential is

$$\begin{aligned} \nabla^2 V_2(\mathbf{\Lambda}) = & \left(\frac{1}{|\mathbf{\Lambda}|^3} + \frac{3}{2}\frac{1}{|\mathbf{\Lambda}|^5} - \frac{15}{2}\frac{\mathbf{\Lambda}\cdot\mathbf{I}\mathbf{\Lambda}}{|\mathbf{\Lambda}|^7} \right)\mathbf{1} + \frac{3}{|\mathbf{\Lambda}|^5}\mathbf{I} \\ & + \left(-\frac{3}{|\mathbf{\Lambda}|^5} - \frac{15}{2}\frac{1}{|\mathbf{\Lambda}|^7} + \frac{105}{2}\frac{\mathbf{\Lambda}\cdot\mathbf{I}\mathbf{\Lambda}}{|\mathbf{\Lambda}|^9} \right)\mathbf{\Lambda}\mathbf{\Lambda}^{\mathsf{T}} - \frac{15}{|\mathbf{\Lambda}|^7}\left(\mathbf{\Lambda}\mathbf{\Lambda}^{\mathsf{T}}\mathbf{I} + \mathbf{I}\mathbf{\Lambda}\mathbf{\Lambda}^{\mathsf{T}} \right). \end{aligned} \tag{7.38}$$

Incorporating the principal relative equilibrium conditions, we find

$$\nabla^2 V_2(\mathbf{\Lambda}_e) =$$

$$\frac{1}{R^5}\begin{bmatrix} R^2 + \frac{3}{2} - \frac{15}{2}I_3 + 3I_1 & 0 & 0 \\ 0 & R^2 + \frac{3}{2} - \frac{15}{2}I_3 + 3I_2 & 0 \\ 0 & 0 & -2R^2 - 6 + 18I_3 \end{bmatrix}. \tag{7.39}$$

Since we have $v_{12} = v_{13} = v_{23} = 0$, the linear system matrix decouples into two subsystems: a pitch system associated with $(\Sigma_1, \Sigma_3, \Lambda_1, \Lambda_3, \Pi_2)$ and a roll-yaw system associated with

$(\Sigma_2, \Lambda_2, \Pi_1, \Pi_3)$. The characteristic equations for these two subsystems are

$$P_p(s) = s(s^4 + A_2 s^2 + A_0) \tag{7.40a}$$

$$P_{ry}(s) = s^4 + B_2 s^2 + B_0 \tag{7.40b}$$

where

$$A_0 = \frac{3}{2I_2R^{10}}(I_1 - I_3)\left[2R^4 - (3 + 6I_2 - 9I_3)R^2 - 15I_2(1 - 3I_3)\right] \tag{7.41a}$$

$$A_2 = \frac{1}{2I_2R^5}\left[2(3I_1 + I_2 - 3I_3)R^2 - 3I_2(I_2 - I_1)\right] \tag{7.41b}$$

$$B_0 = \frac{1}{4I_1I_3R^{10}}(I_2 - I_1)(I_2 - I_3)(2R^2 + 3 - 9I_3)(8R^2 + 3 + 6I_2 - 15I_3) \tag{7.41c}$$

$$\begin{aligned} B_2 = \frac{1}{2I_1I_3R^5} & \left[(-2I_1I_2 + 2I_2^2 + 4I_1I_3 + 4I_2I_3 - 6I_3^2)R^2 \right. \\ & \left. + (15I_1I_2I_3 + 6I_1^2I_3 - 18I_1I_3^2 + 3I_2^3 - 9I_2^2I_3 - 3I_2I_3^2)\right]. \end{aligned} \tag{7.41d}$$

The spectral stability conditions given in Table 3.1 are

$$A_0 \geq 0 \quad A_2 \geq 0 \quad A_2^2 - 4A_0 \geq 0 \tag{7.42a}$$

$$B_0 \geq 0 \quad B_2 \geq 0 \quad B_2^2 - 4A_0 \geq 0 \tag{7.42b}$$

which, upon substitution of the inertia parameters for the inertias, may be reduced to

$$(k_1 - k_3)\left[2f_1^2 + 3f_1f_2\left(\frac{1}{R}\right)^2 + 15(1 - k_1k_3)f_3\left(\frac{1}{R}\right)^4\right] \geq 0 \tag{7.43a}$$

$$2f_1f_4 - 3k_3(1 - k_1)(1 - k_1k_3)\left(\frac{1}{R}\right)^2 \geq 0 \tag{7.43b}$$

$$4f_1^2f_5^2 - 12f_1f_6(1 - k_1k_3)\left(\frac{1}{R}\right)^2 + 9f_7(1 - k_1k_3)^2\left(\frac{1}{R}\right)^4 \geq 0 \tag{7.43c}$$

$$k_1k_3\left[16f_1^2 + 6f_1f_8\left(\frac{1}{R}\right)^2 + 9f_3f_9\left(\frac{1}{R}\right)^4\right] \geq 0 \tag{7.43d}$$

$$2f_1(1 + 3k_1 + k_1k_3) + 3f_{10}\left(\frac{1}{R}\right)^2 \geq 0 \tag{7.43e}$$

$$4f_1^2 f_{11} + 12 f_1 f_{12} \left(\frac{1}{R}\right)^2 + 9 f_{13}^2 \left(\frac{1}{R}\right)^4 \geq 0. \tag{7.43f}$$

where

$$f_1 = 3 - k_1 - k_3 - k_1 k_3 \qquad f_2 = -2 - 2k_1 + k_3 + 3k_1 k_3 \qquad f_3 = -2k_1 + k_3 + k_1 k_3$$

$$f_4 = 1 + 3k_1 - 3k_3 - k_1 k_3 \qquad f_5 = 1 - 3k_1 + 3k_3 - k_1 k_3$$

$$f_6 = -12k_1 - 12k_1^2 + 13k_3 + 20k_1 k_3 + 15k_1^2 k_3 - 9k_3^2 - 16k_1 k_3^2 + k_1^2 k_3^2$$

$$f_7 = 80k_1^2 - 120k_1 k_3 - 40k_1^2 k_3 + 41k_3^2 + 38k_1 k_3^2 + k_1^2 k_3^2$$

$$f_8 = 12k_1 - 5k_3 - 7k_1 k_3 \qquad f_9 = -4k_1 + k_3 + 3k_1 k_3$$

$$f_{10} = 4k_1 - k_3 - 3k_1 k_3 + 2k_1^2 k_3 - k_1 k_3^2 - k_1^2 k_3^2$$

$$f_{11} = 1 + 6k_1 + 9k_1^2 - 14k_1 k_3 + 6k_1^2 k_3 + k_1^2 k_3^2$$

$$f_{12} = 4k_1 + 12k_1^2 - k_3 - 6k_1 k_3 - 27k_1^2 k_3 + 6k_1^3 k_3 + 8k_1 k_3^2 + 7k_1^2 k_3^2 - k_1^3 k_3^2 - k_1^2 k_3^3 - k_1^3 k_3^3$$

$$f_{13} = 4k_1 - k_3 - 3k_1 k_3 - 2k_1^2 k_3 + k_1 k_3^2 + k_1^2 k_3^2$$

In contrast to the Keplerian system, spectral stability of the free system is dependent on the radius. However, note that we have purposely arranged the stability conditions in powers of $(1/R)^2$. For practical applications (with $R \gg 10$), we may neglect all but the zeroth-order terms. The resulting conditions reduce to precisely the conditions for spectral stability of the Keplerian system. This approach is valid as long as the terms we keep do not approach zero, which occurs in the vicinity of the curves shown in Figure 6.2 (typically when two inertias are nearly equal).

7.1.5 Nonlinear Stability (2nd-Order Approximation). The Hessian of the variational Lagrangian was given in Equation (7.15) and, at relative equilibrium, reduces to

$$\nabla^2 F(\mathbf{z}_e) =$$
$$\begin{bmatrix} \frac{I_2}{\beta} & 0 & 0 & 0 & 0 & \frac{-\omega(\beta+R^2)}{\beta} & 0 & \frac{-R}{\beta} & 0 \\ 0 & \frac{I_2}{\beta} & 0 & 0 & 0 & 0 & \frac{R}{\beta} & 0 & 0 \\ 0 & 0 & 1 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & \frac{3+6I_1-15I_3+2R^2}{2R^5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_{55} & 0 & 0 & 0 & \frac{\omega R}{\beta} \\ \frac{-\omega(\beta+R^2)}{\beta} & 0 & 0 & 0 & 0 & f_{66} & 0 & \frac{-\omega R}{\beta} & 0 \\ 0 & \frac{R}{\beta} & 0 & 0 & 0 & 0 & \frac{-I_1+I_2+R^2}{I_1\beta} & 0 & 0 \\ \frac{-R}{\beta} & 0 & 0 & 0 & 0 & \frac{-\omega R}{\beta} & 0 & \frac{R^2}{I_2\beta} & 0 \\ 0 & 0 & 0 & 0 & \frac{\omega R}{\beta} & 0 & 0 & 0 & \frac{I_2-I_3+R^2}{I_3\beta} \end{bmatrix} \tag{7.44}$$

where

$$f_{55} = \frac{3I_2 + 6I_2^2 - 15I_2I_3 + 8I_2R^2 - 6I_3R^2}{2R^5\beta}$$
$$f_{66} = -\frac{12I_2 - 36I_2I_3 + 15R^2 + 4I_2R^2 - 45I_3R^2 + 6R^4)}{2R^5\beta}$$

and $\beta = I_2 + R^2$. We can show that this matrix is indefinite, so we apply the projection method to examine nonlinear stability of this system. Recall, the projection onto the tangent space is given by

$$P(\mathbf{z}_e) = \mathbf{1} - \mathbf{K}(\mathbf{z}_e)\left[\mathbf{K}^{\mathsf{T}}(\mathbf{z}_e)\mathbf{K}(\mathbf{z}_e)\right]^{-1}\mathbf{K}^{\mathsf{T}}(\mathbf{z}_e) \tag{7.45}$$

where $\mathbf{K}(\mathbf{z})$ is the matrix whose columns form a basis for the left nullspace of $\mathbf{A}(\mathbf{z})$. For the current problem, the nullspace is one-dimensional since there is only one Casimir function:

$$\mathbf{K}(\mathbf{z}_e) = \begin{bmatrix} R & 0 & 0 & 0 & 0 & \omega R & 0 & 1 & 0 \end{bmatrix}^{\mathsf{T}}. \tag{7.46}$$

The projected Hessian may then be computed as

$$\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e) =$$

$$\begin{bmatrix} f_{11} & 0 & 0 & 0 & 0 & f_{16} & 0 & f_{18} & 0 \\ 0 & \frac{I_2}{\beta} & 0 & 0 & 0 & 0 & \frac{R}{\beta} & 0 & 0 \\ 0 & 0 & 1 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & \frac{3+6I_1-15I_3+2R^2}{2R^5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_{55} & 0 & 0 & 0 & \frac{\omega R}{\beta} \\ f_{16} & 0 & 0 & 0 & 0 & f_{66} & 0 & f_{68} & 0 \\ 0 & \frac{R}{\beta} & 0 & 0 & 0 & 0 & \frac{-I_1+I_2+R^2}{I_1\beta} & 0 & 0 \\ f_{18} & 0 & 0 & 0 & 0 & f_{68} & 0 & f_{88} & 0 \\ 0 & 0 & 0 & 0 & \frac{\omega R}{\beta} & 0 & 0 & 0 & \frac{I_2-I_3+R^2}{I_3\beta} \end{bmatrix} \tag{7.47}$$

where

$$\begin{aligned} f_{11} &= \frac{1}{d}(-9I_2 + 54I_2I_3 - 81I_2I_3^2 + 24I_2R^3 - 72I_2I_3R^3 + 4I_2R^4 + 16I_2R^5 + 4I_2R^6 + 4R^8) \\ f_{16} &= \frac{2\omega R^3}{d}(9I_2 - 27I_2I_3 + 5I_2R^2 - 9I_2I_3R^2 - 2I_2R^3 - 2I_2R^4 + 2R^5 - 4I_2R^5) \\ f_{18} &= \frac{1}{d}(-27I_2 + 162I_2I_3 - 243I_2I_3^2 - 24I_2R^2 + 72I_2I_3R^2 + 6I_2R^3 - 18I_2I_3R^3 - 4I_2R^4 \\ &\qquad - 6R^5 - 8I_2R^5 + 18I_3R^5 + 36I_2I_3R^5 - 4R^7 - 8I_2R^7 - 4I_2R^8 - 4R^{10}) \\ f_{66} &= \frac{2R}{d}(-12I_2 + 36I_2I_3 + 3R^2 - 22I_2R^2 - 9I_3R^2 \\ &\qquad + 54I_2I_3R^2 + 2R^4 - 7I_2R^4 + 9I_2I_3R^4 + 2I_2R^6) \\ f_{68} &= \frac{2\omega R^2}{d}(12I_2 - 36I_2I_3 - 3R^2 + 13I_2R^2 + 9I_3R^2 \\ &\qquad - 27I_2I_3R^2 - 2R^4 + 2I_2R^4 + 2I_2R^5 - 2R^7 + 4I_2R^7) \\ f_{88} &= \frac{1}{d}(-36I_2 + 216I_2I_3 - 324I_2I_3^2 + 9R^2 - 36I_2R^2 - 54I_3R^2 \\ &\qquad + 108I_2I_3R^2 + 81I_3^2R^2 + 12R^4 - 8I_2R^4 - 36I_3R^4 - 12I_2R^5 + 36I_2I_3R^5 \\ &\qquad + 4R^6 + 12R^7 - 8I_2R^7 - 36I_3R^7 + 8R^9 + 4I_2R^{10} + 4R^{12}) \\ d &= I_2(3 - 9I_3 + 2R^2 + 2R^3 + 2R^5)^2 \end{aligned}$$

and f_{55} is as defined for the original Hessian.

Analysis of the matrix is simplified by the fact that it may be put into block-diagonal form by the permutation which reorders the phase variables as

$$(\Sigma_1, \Lambda_3, \Pi_2, \Sigma_2, \Pi_1, \Sigma_3, \Lambda_1, \Lambda_2, \Pi_3)\,.$$

The result is a 3×3 block in the upper left corner, followed by three separate 2×2 blocks.[3] In order to determine the signs of the eigenvalues, we factor this symmetric matrix into the form LDL^{T} where L is lower triangular with ones on the diagonal and D is a diagonal matrix of pivots. By Sylvester's law of inertia,[4] the signs of the pivots in D agree with the signs of the eigenvalues of $\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)$. One pivot in the 3×3 block is zero and is associated with the Casimir constraint. The remaining eight are associated with the tangent space. Eliminating factors which are strictly positive for $R > 0$, the conditions for positive definiteness are:[5]

$$\begin{aligned} 4 + 4I_2\left(\frac{1}{R}\right)^2 + 16I_2\left(\frac{1}{R}\right)^3 + 4I_2\left(\frac{1}{R}\right)^4 \\ + 24I_2(1-3I_3)\left(\frac{1}{R}\right)^5 - 9I_2(1-3I_3)^2\left(\frac{1}{R}\right)^8 > 0 \end{aligned} \tag{7.48a}$$

$$\begin{aligned} \left[2 + 2\left(\frac{1}{R}\right)^2 + 2\left(\frac{1}{R}\right)^3 + 3(1-3I_3)\left(\frac{1}{R}\right)^5\right]^2 \\ \cdot \left[2 - 3(1+2I_2-3I_3)\left(\frac{1}{R}\right)^2 - 15I_2(1-3I_3)\left(\frac{1}{R}\right)^4\right] > 0 \end{aligned} \tag{7.48b}$$

$$(I_2 - I_1) > 0 \tag{7.48c}$$

$$(I_1 - I_3) > 0 \tag{7.48d}$$

$$2(4I_2 - 3I_3) + 3I_2(1 + 2I_2 - 5I_3)\left(\frac{1}{R}\right)^2 > 0 \tag{7.48e}$$

[3]The same approach was used to analyze the variational Lagrangian prior to projection. The only portion of the matrix changed by the projection is the 3×3 block associated with $(\Sigma_1, \Lambda_3, \Pi_2)$.

[4]See, *e.g.*, Strang [98:341].

[5]Some of the pivots are strictly positive, so negative definiteness, which would also prove nonlinear stability, is not possible.

$$(I_2 - I_3)\left[8 + 3(1 + 2I_2 - 5I_3)\left(\frac{1}{R}\right)^2\right] > 0. \tag{7.48f}$$

The conditions given by Equations (7.48c) and (7.48d) restrict our results to configurations in the Lagrange region ($I_2 > I_1 > I_3$) of the inertia parameter plane. With this restriction, Equations (7.48e) and (7.48f) are satisfied for any value of R (recognizing that by the nondimensionalization of the inertia matrix we have $1/3 < I_2 \leq 1/2$ and $0 < I_3 < 1/3$ in the Lagrange region). Likewise, the lead term in Equations (7.48b) is strictly positive and can be factored out of this equation. It is clear that for large R the remaining conditions are satisfied. We find that the lower bound on R is given by Equation (7.48b) as

$$R^2 > \frac{1}{4}\left(6 + \sqrt{96}\right) \approx 3.95\,. \tag{7.49}$$

Thus, for all practical radii, the Lagrange region is nonlinearly stable. By implication, the spectral stability conditions must also be satisfied under these circumstances.

7.2 Relative Equilibria for an Axisymmetric Rigid Body

7.2.1 Transformation to Nodal Frame (General Development). For an axisymmetric body, the angular momentum about the symmetry axis is conserved. For the free rigid-body system the integral is

$$C_2(\mathbf{z}) = \mathbf{\Pi} \cdot \mathbf{1}_3 \tag{7.50}$$

where we assume the symmetry axis is directed along the third basis vector in the body frame. Just as in the Keplerian system, we treat this problem by transforming to the nodal frame by a rotation about the symmetry axis. Following the arguments presented in Section 6.2.1, the axisymmetric Hamiltonian system is the Hamiltonian system of (7.3), but with

$$H(\mathbf{z}) = H_b(\mathbf{z}) - \mu_2 C_2(\mathbf{z}). \tag{7.51}$$

Here H_b is the Hamiltonian for the arbitrary rigid body system (expressed in nodal frame variables) and μ_2 is the spin rate of the body frame relative to the nodal frame. The equations of motion $\dot{\mathbf{z}}_e = \mathbf{J}(\mathbf{z})\nabla H(\mathbf{z})$ take the form

$$\begin{pmatrix} \dot{\mathbf{\Sigma}} \\ \dot{\mathbf{\Lambda}} \\ \dot{\mathbf{\Pi}} \end{pmatrix} = \begin{bmatrix} \mathbf{0} & -\mathbf{1} & \mathbf{\Sigma}^\times \\ \mathbf{1} & \mathbf{0} & \mathbf{\Lambda}^\times \\ \mathbf{\Sigma}^\times & \mathbf{\Lambda}^\times & \mathbf{\Pi}^\times \end{bmatrix} \begin{pmatrix} \mathbf{\Sigma} \\ \nabla V(\mathbf{\Lambda}) \\ \mathbf{I}^{-1}\mathbf{\Pi} - \mu_2 \mathbf{1}_3 \end{pmatrix} \tag{7.52}$$

The only change from the equations for the arbitrary body system is in the last term of $\nabla H(\mathbf{z})$. In addition, since the symmetry integral is linear in the phase variables, it does not affect the Hessian of the variational Lagrangian which will be presented in the next section. Hence, the form of the Hessian given for the arbitrary body in Equation (7.15) and the form of the linear system matrix given in Equation (7.16) still apply to the axisymmetric system.

7.2.2 Relative Equilibrium Conditions (General Development). The relative equilibria for the axisymmetric system are critical points of the variational Lagrangian

$$F(\mathbf{z}) = H(\mathbf{z}) - \mu_1 C_1(\mathbf{z}) = H_b(\mathbf{z}) - \mu_1 C_1(\mathbf{z}) - \mu_2 C_2(\mathbf{z}). \tag{7.53}$$

Taking the gradient gives the relative equilibrium conditions

$$\mathbf{\Sigma}_e + \mu_1 \mathbf{\Lambda}_e{}^\times (\mathbf{\Pi}_e + \mathbf{\Lambda}_e{}^\times \mathbf{\Sigma}_e) = \mathbf{0} \tag{7.54a}$$

$$\nabla V(\mathbf{\Lambda}_e) - \mu_1 \mathbf{\Sigma}_e{}^\times (\mathbf{\Pi}_e + \mathbf{\Lambda}_e{}^\times \mathbf{\Sigma}_e) = \mathbf{0} \tag{7.54b}$$

$$\mathbf{I}^{-1}\mathbf{\Pi}_e - \mu_1 (\mathbf{\Pi}_e + \mathbf{\Lambda}_e{}^\times \mathbf{\Sigma}_e) - \mu_2 \mathbf{1}_3 = \mathbf{0}. \tag{7.54c}$$

The solutions to these equations are parameterized by the Casimir and symmetry integral conditions

$$C_1 = \frac{1}{2} \left| \mathbf{\Pi}_e + \mathbf{\Lambda}_e{}^{\times} \mathbf{\Omega}_e \right|^2 \tag{7.55a}$$

$$C_2 = \mathbf{1}_3 \cdot \mathbf{\Pi}_e \tag{7.55b}$$

where C_1 and C_2 without the arguments represent constants. Similar to the treatment for the arbitrary body, Equation (7.54) may be expressed as (*cf.* Equation (7.8))

$$\mathbf{\Sigma}_e + \mu_1 \mathbf{\Lambda}_e{}^{\times} \mathbf{\Gamma}_e = \mathbf{0} \tag{7.56a}$$

$$\nabla V(\mathbf{\Lambda}_e) - \mu_1 \mathbf{\Sigma}_e{}^{\times} \mathbf{\Gamma}_e = \mathbf{0} \tag{7.56b}$$

$$\widetilde{\mathbf{\Omega}}_e - \mu_1 \mathbf{\Gamma}_e = \mathbf{0} \tag{7.56c}$$

where, for convenience, we have introduced the angular momentum about the center of mass, $\mathbf{\Gamma} = \mathbf{\Pi} + \mathbf{\Lambda}^{\times}\mathbf{\Sigma}$, and the angular velocity of the nodal frame, $\widetilde{\mathbf{\Omega}} = \mathbf{\Omega} - \mu_2 \mathbf{1}_3$. By Equation (7.56c), Equation (7.56a) and (7.56b) reduce to

$$\mathbf{\Sigma}_e + \mathbf{\Lambda}_e{}^{\times} \widetilde{\mathbf{\Omega}}_e = \mathbf{0} \tag{7.57a}$$

$$\nabla V(\mathbf{\Lambda}_e) - \mathbf{\Sigma}_e{}^{\times} \widetilde{\mathbf{\Omega}}_e = \mathbf{0}. \tag{7.57b}$$

Using Equation (7.57a), we may eliminate $\mathbf{\Sigma}_e$ so that the relative equilibrium conditions expressed in terms of $\mathbf{\Lambda}_e$, $\widetilde{\mathbf{\Omega}}_e$, μ_1, and μ_2 are

$$\nabla V(\mathbf{\Lambda}_e) + \widetilde{\mathbf{\Omega}}_e{}^{\times} \widetilde{\mathbf{\Omega}}_e{}^{\times} \mathbf{\Lambda}_e = \mathbf{0} \tag{7.58a}$$

$$\widetilde{\mathbf{\Omega}}_e - \mu_1 \left(\mathbf{I} \widetilde{\mathbf{\Omega}}_e - \mathbf{\Lambda}_e{}^{\times} \mathbf{\Lambda}_e{}^{\times} \widetilde{\mathbf{\Omega}}_e + \mu_2 \mathbf{I} \mathbf{1}_3 \right) = \mathbf{0}. \tag{7.58b}$$

Furthermore, the Casimir and symmetry integral constraints given in Equation (7.55) become

$$C_1 = \frac{1}{2}\left|\frac{\widetilde{\mathbf{\Omega}}_e}{\mu_1}\right|^2 \tag{7.59a}$$

$$C_2 = \mathbf{1}_3 \cdot \mathbf{I}\left(\widetilde{\mathbf{\Omega}}_e + \mu_2 \mathbf{1}_3\right). \tag{7.59b}$$

Introducing $\beta = 1/\mu_1$, $c_1 = \mu_1^2 C_1$, and $c_2 = C_2/I_a$, Equations (7.58) and (7.59) may be written in a form similar to that given by Wang *et al* [105] for the arbitrary body:

$$\nabla V(\mathbf{\Lambda}_e) + \widetilde{\mathbf{\Omega}}_e{}^{\times}\widetilde{\mathbf{\Omega}}_e{}^{\times}\mathbf{\Lambda}_e = \mathbf{0} \tag{7.60a}$$

$$\beta\widetilde{\mathbf{\Omega}}_e - \mathbf{I}\widetilde{\mathbf{\Omega}}_e + \mathbf{\Lambda}_e{}^{\times}\mathbf{\Lambda}_e{}^{\times}\widetilde{\mathbf{\Omega}}_e - \mu_2 \mathbf{I}\mathbf{1}_3 = \mathbf{0} \tag{7.60b}$$

$$c_1 - \frac{1}{2}\widetilde{\mathbf{\Omega}}_e \cdot \widetilde{\mathbf{\Omega}}_e = 0 \tag{7.60c}$$

$$c_2 - \mathbf{1}_3 \cdot \widetilde{\mathbf{\Omega}}_e - \mu_2 = 0. \tag{7.60d}$$

The last condition allows us to parameterize by μ_2 in place of c_2. The first three conditions are equivalent to the variational problem

$$\begin{gathered}\text{Make stationary } L(\mathbf{\Lambda}, \widetilde{\mathbf{\Omega}}) = \frac{1}{2}\left(\widetilde{\mathbf{\Omega}} + \mu_2 \mathbf{1}_3\right) \cdot \mathbf{I}\left(\widetilde{\mathbf{\Omega}} + \mu_2 \mathbf{1}_3\right) + \frac{1}{2}\left|\widetilde{\mathbf{\Omega}}^{\times}\mathbf{\Lambda}\right| - V(\mathbf{\Lambda}) \\ \text{subject to } \frac{1}{2}\widetilde{\mathbf{\Omega}} \cdot \widetilde{\mathbf{\Omega}} = c_1.\end{gathered} \tag{7.61}$$

Again, this is different from the variational characterization discussed in Chapter 3 related to the variational Lagrangian. Equations (7.60a)–(7.60c) form a nonlinear system of seven equations in seven unknowns ($\widetilde{\mathbf{\Omega}}_e$, $\mathbf{\Lambda}_e$, and β) parameterized by c_1 and μ_2.

The form of Equations (7.60a)–(7.60c) is similar to that of Equations (7.12a)–(7.12c) with the nodal frame angular velocity replacing the body frame angular velocity. We follow the same procedure as for the arbitrary body which lead to general remarks concerning the nature of relative equilibria. Taking the dot products and cross products of $\mathbf{\Lambda}_e$ and

$\widetilde{\mathbf{\Omega}}_e$ with Equation (7.60a) gives

$$\mathbf{\Lambda}_e \cdot \nabla V(\mathbf{\Lambda}_e) = |\widetilde{\mathbf{\Omega}}_e|^2 \, |\mathbf{\Lambda}_e|^2 - (\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Lambda}_e)^2 \tag{7.62a}$$

$$\mathbf{\Lambda}_e{}^{\times} \nabla V(\mathbf{\Lambda}_e) = (\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Lambda}_e) \widetilde{\mathbf{\Omega}}_e{}^{\times} \mathbf{\Lambda}_e \tag{7.62b}$$

$$\widetilde{\mathbf{\Omega}}_e \cdot \nabla V(\mathbf{\Lambda}_e) = 0 \tag{7.62c}$$

$$\widetilde{\mathbf{\Omega}}_e{}^{\times} \nabla V(\mathbf{\Lambda}_e) = |\widetilde{\mathbf{\Omega}}_e|^2 \, \widetilde{\mathbf{\Omega}}_e{}^{\times} \mathbf{\Lambda}_e. \tag{7.62d}$$

Likewise, the dot and cross products of $\mathbf{\Lambda}_e$ and $\widetilde{\mathbf{\Omega}}_e$ with Equation (7.60b) reduce to

$$\mathbf{\Lambda}_e \cdot \mathbf{I}\widetilde{\mathbf{\Omega}}_e = \beta(\mathbf{\Lambda}_e \cdot \widetilde{\mathbf{\Omega}}_e) - \mu_2 \mathbf{\Lambda}_e \cdot \mathbf{I1}_3 \tag{7.63a}$$

$$\mathbf{\Lambda}_e{}^{\times} \mathbf{I}\widetilde{\mathbf{\Omega}}_e = (\beta - |\mathbf{\Lambda}_e|^2) \mathbf{\Lambda}_e{}^{\times} \widetilde{\mathbf{\Omega}}_e - \mu_2 \mathbf{\Lambda}_e{}^{\times} \mathbf{I1}_3 \tag{7.63b}$$

$$\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{I}\widetilde{\mathbf{\Omega}}_e = (\beta - |\mathbf{\Lambda}_e|^2) \, |\widetilde{\mathbf{\Omega}}_e|^2 + (\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Lambda}_e)^2 - \mu_2 \widetilde{\mathbf{\Omega}}_e \cdot \mathbf{I1}_3 \tag{7.63c}$$

$$\widetilde{\mathbf{\Omega}}_e{}^{\times} \mathbf{I}\widetilde{\mathbf{\Omega}}_e = (\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Lambda}_e) \widetilde{\mathbf{\Omega}}_e{}^{\times} \mathbf{\Lambda}_e - \mu_2 \widetilde{\mathbf{\Omega}}_e{}^{\times} \mathbf{I1}_3. \tag{7.63d}$$

To emphasize the geometric implications more clearly, these last four equations may be rewritten as

$$\mathbf{\Lambda}_e \cdot \mathbf{\Pi}_e = \beta(\mathbf{\Lambda}_e \cdot \widetilde{\mathbf{\Omega}}_e) \tag{7.64a}$$

$$\mathbf{\Lambda}_e{}^{\times} \mathbf{\Pi}_e = (\beta - |\mathbf{\Lambda}_e|^2) \mathbf{\Lambda}_e{}^{\times} \widetilde{\mathbf{\Omega}}_e \tag{7.64b}$$

$$\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Pi}_e = (\beta - |\mathbf{\Lambda}_e|^2) \, |\widetilde{\mathbf{\Omega}}_e|^2 + (\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Lambda}_e)^2 \tag{7.64c}$$

$$\widetilde{\mathbf{\Omega}}_e{}^{\times} \mathbf{\Pi}_e = (\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Lambda}_e) \widetilde{\mathbf{\Omega}}_e{}^{\times} \mathbf{\Lambda}_e. \tag{7.64d}$$

Equations (7.62) and (7.64) are identical to Equations (7.17) and (7.18) with $\widetilde{\mathbf{\Omega}}_e$ in place of $\mathbf{\Omega}_e$ and $\mathbf{\Pi}_e$ in place of $\mathbf{I\Omega}_e$. However, whereas we were assured $\mathbf{\Omega}_e \cdot \mathbf{I\Omega}_e$ is positive, the same is not necessarily true for $\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Pi}_e$. Therefore, the remarks made for the arbitrary body need to be reconsidered.

Remark 1(s). Solving for β in Equation (7.63c) gives

$$\beta = \frac{1}{|\widetilde{\mathbf{\Omega}}_e|^2}\left[\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{I}\widetilde{\mathbf{\Omega}}_e + \mu_2\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{I}\mathbf{1}_3 + |\mathbf{\Lambda}_e|^2 |\widetilde{\mathbf{\Omega}}_e|^2 - (\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Lambda}_e)^2\right]. \tag{7.65}$$

Previously, we found β to be positive. Now it is only guaranteed to be positive when the dot product of $\widetilde{\mathbf{\Omega}}_e$ with $\mu_2\mathbf{1}_3$ is positive. Note that $\widetilde{\mathbf{\Omega}}_e$ is the rotation due to orbital motion while $\mu_2\mathbf{1}_3$ is the rotation due to spin about the symmetry axis. When the dot product is positive, we will refer to this as a positively oriented spin.

Remark 2(s). By consideration of Equations (7.57a) and (7.57b), we find that the vectors $\mathbf{\Sigma}_e$, $\widetilde{\mathbf{\Omega}}_e$, and $\mathbf{\Lambda}_e$ are an orthogonal triplet. By combining Equations (7.62b) and (7.63d), we have the Pringle-Likins Condition for the free rigid body:

$$\mathbf{\Lambda}_e{}^{\times}\nabla V(\mathbf{\Lambda}_e) - \widetilde{\mathbf{\Omega}}_e{}^{\times}\mathbf{I}\widetilde{\mathbf{\Omega}}_e - \mu_2\widetilde{\mathbf{\Omega}}_e{}^{\times}\mathbf{I}\mathbf{1}_3 = \mathbf{0}. \tag{7.66}$$

Written as

$$\mathbf{\Lambda}_e{}^{\times}\nabla V(\mathbf{\Lambda}_e) - \widetilde{\mathbf{\Omega}}_e{}^{\times}\mathbf{\Pi}_e = \mathbf{0} \tag{7.67}$$

it becomes clear that $\mathbf{\Pi}_e$ and $\mathbf{\Lambda}_e$ must be in the plane formed by $\widetilde{\mathbf{\Omega}}_e$ and $\nabla V(\mathbf{\Lambda}_e)$. By the Cauchy-Schwarz inequality, $\mathbf{\Lambda}_e \cdot \nabla V(\mathbf{\Lambda}_e)$ is positive; however, as noted above, $\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Pi}_e$ can be either positive or negative. We find that in contrast to the three possible configurations for the arbitrary body, we have seventeen configurations as shown in Figures 7.3 and 7.4. These figures show the direction of $\widetilde{\mathbf{\Omega}}_e$, $\mathbf{\Lambda}_e$, $\mathbf{\Pi}_e$, and $\nabla V(\mathbf{\Lambda}_e)$ when looking in the $-\mathbf{\Sigma}_e$ direction. The direction of $\mathbf{\Lambda}_e$ is representative based on the sign of $\mathbf{\Lambda}_e \cdot \widetilde{\mathbf{\Omega}}_e$. The arc represents the range of possible directions for $\mathbf{\Pi}_e$ corresponding to different values of β as given in Table 7.1. The parameter β_0 in the table is

$$\beta_0 = \frac{1}{|\widetilde{\mathbf{\Omega}}_e|^2}\left[|\widetilde{\mathbf{\Omega}}_e|^2 |\mathbf{\Lambda}_e|^2 - (\widetilde{\mathbf{\Omega}}_e \cdot \boldsymbol{\lambda}_e)^2\right]. \tag{7.68}$$

Table 7.1 Relative Equilibrium Configurations

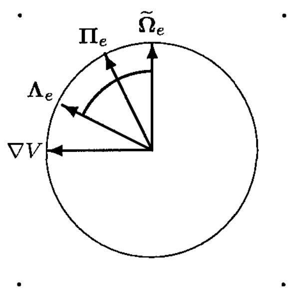
	$\beta > \lvert\mathbf{\Lambda}_e\rvert^2$	$\beta = \lvert\mathbf{\Lambda}_e\rvert^2$	$\beta_0 < \beta < \lvert\mathbf{\Lambda}_e\rvert^2$	$\beta = \beta_0$	$0 < \beta < \beta_0$	$\beta = 0$	$\beta < 0$
$\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Lambda}_e > 0$	1	2	3	4	5	6	7
$\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Lambda}_e = 0$	8	8	8	8	8	9	10
$\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Lambda}_e < 0$	11	12	13	14	15	16	17

Recognize that these configurations are only possible classes of relative equilibria. Just as in the second-order arbitrary body system where only the orthogonal class existed for $R > 3/2$, the presence of these equilibria is not certain. In addition, we are unable to specify the direction of the symmetry axis based upon these results. Further analysis requires more information about the potential function.

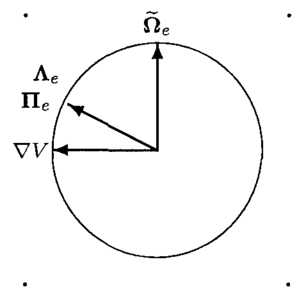
Remark 3(s). If we assume the orbit is orthogonal ($\widetilde{\mathbf{\Omega}}_e \cdot \mathbf{\Lambda}_e = 0$), then we find that the force is aligned with the radial direction ($\nabla V(\mathbf{\Lambda}_e)$ and $\mathbf{\Lambda}_e$ are parallel), the angular momentum is aligned with the angular velocity ($\mathbf{\Pi}_e$ and $\widetilde{\mathbf{\Omega}}_e$ are parallel), and the gravitational torque is identically zero ($\mathbf{\Lambda}_e{}^\times \nabla V(\mathbf{\Lambda}_e)$ is zero). The argument is reversible for each of these properties, so that they are unique to orthogonal relative equilibria. However, we no longer conclude the rotation is about a principal axis as for the arbitrary body.

Remark 4(s). Recognizing that the orbital angular velocity is now the nodal frame angular velocity, not the body frame angular velocity, the generalized Keplerian frequency is still given by Equation (7.25) with $\widetilde{\mathbf{\Omega}}_e$ substituted for $\mathbf{\Omega}_e$. This can be derived directly from Equation (7.62a).

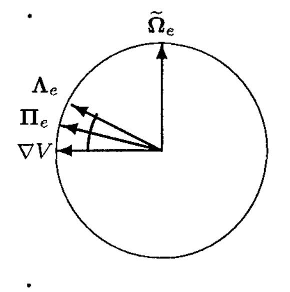
7.2.3 Relative Equilibrium Classes (2nd-Order Approximation). We begin our analysis of the relative equilibria for the second-order approximation by determining classes of solutions to the Pringle-Likins Condition. We then substitute these solutions into the full relative equilibrium conditions of Equation (7.60) to determine solutions for all the variables. For the second-order approximation, the gradient of the potential may be expressed



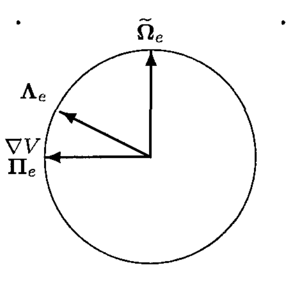
(a) Configuration 1



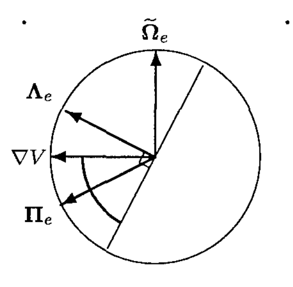
(b) Configuration 2



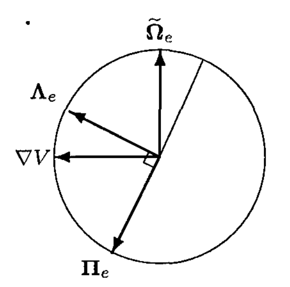
(c) Configuration 3



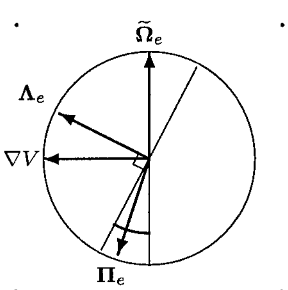
(d) Configuration 4



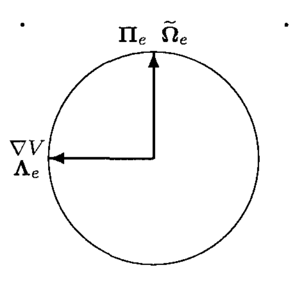
(e) Configuration 5



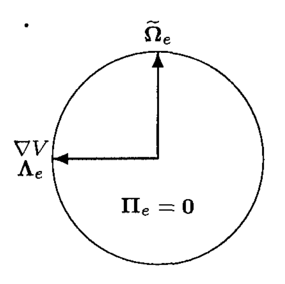
(f) Configuration 6



(g) Configuration 7

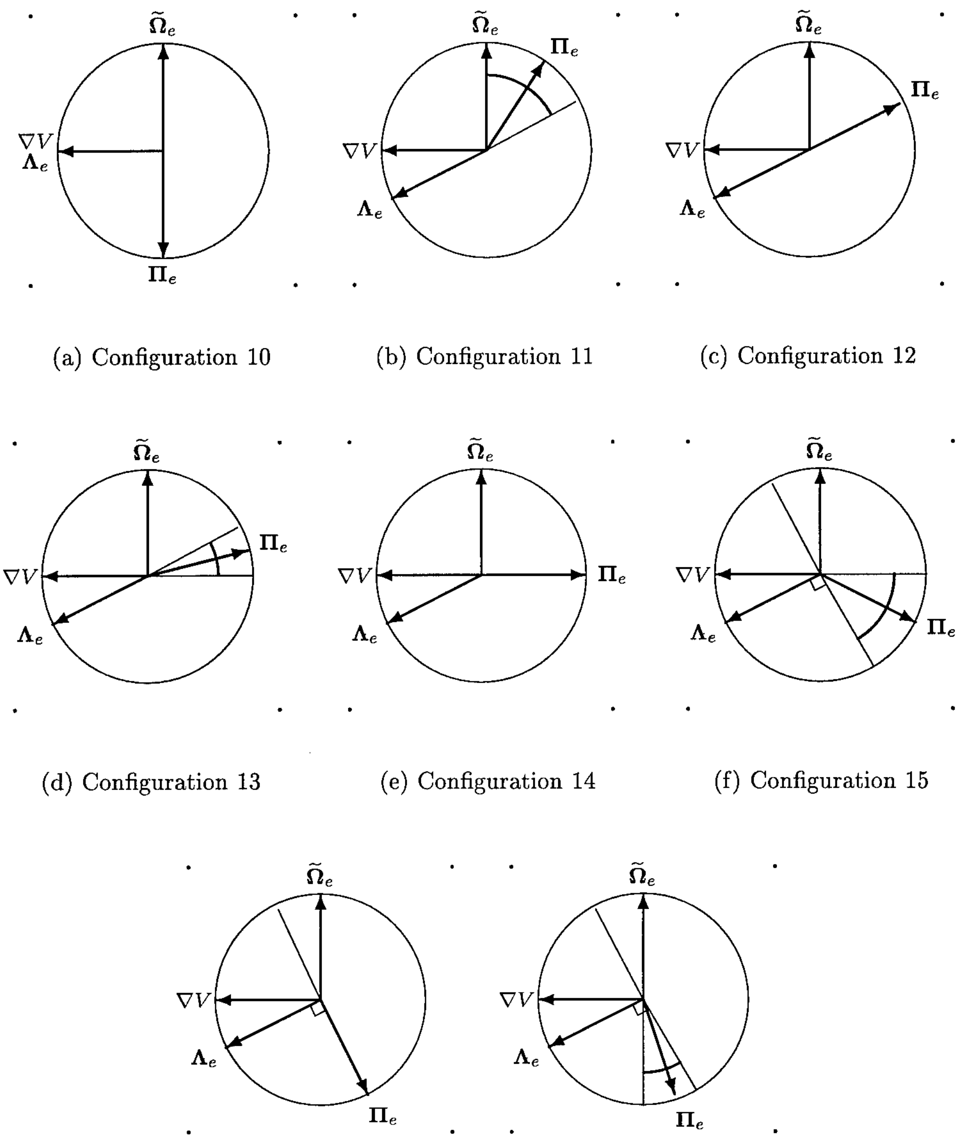


(h) Configuration 8



(i) Configuration 9

Figure 7.3 Relative Equilibrium Geometry for the Axisymmetric Free Rigid Body (Part 1)



(a) Configuration 10 (b) Configuration 11 (c) Configuration 12

(d) Configuration 13 (e) Configuration 14 (f) Configuration 15

(g) Configuration 16 (h) Configuration 17

Figure 7.4 Relative Equilibrium Geometry for the Axisymmetric Free Rigid Body (Part 2)

as

$$\nabla V_2(\boldsymbol{\Lambda}_e) = a_1 \boldsymbol{\Lambda}_e + a_2 \mathbf{I} \boldsymbol{\Lambda}_e \tag{7.69}$$

where

$$a_1 = \frac{1}{|\boldsymbol{\Lambda}_e|^3} + \frac{3}{2}\frac{1}{|\boldsymbol{\Lambda}_e|^5} - \frac{15}{2}\frac{\boldsymbol{\Lambda}_e \cdot \mathbf{I}\boldsymbol{\Lambda}_e}{|\boldsymbol{\Lambda}_e|^7} \qquad a_2 = \frac{3}{|\boldsymbol{\Lambda}_e|^5}.$$

Introducing $b_t = a_1 + a_2 I_t$ and $b_a = a_1 + a_2 I_a$, we may also express the gradient of the potential as $\nabla V_2(\boldsymbol{\Lambda}_e) = (b_t \Lambda_1, b_t \Lambda_2, b_a \Lambda_3)$. With this approximation, the Pringle-Likins Condition given in Equation (7.66) becomes

$$a_2 \boldsymbol{\Lambda}_e{}^{\times} \mathbf{I} \boldsymbol{\Lambda}_e - \widetilde{\boldsymbol{\Omega}}_e{}^{\times} \mathbf{I} \widetilde{\boldsymbol{\Omega}}_e - \mu_2 \widetilde{\boldsymbol{\Omega}}_e{}^{\times} \mathbf{I} \mathbf{1}_3 = \mathbf{0} \tag{7.70}$$

or, in scalar form,

$$a_2 k_t \Lambda_i \Lambda_3 - k_t \widetilde{\Omega}_i \widetilde{\Omega}_3 + Q \widetilde{\Omega}_i = 0 \qquad (i = i, 2). \tag{7.71}$$

Here, we have reintroduced the axisymmetric inertia parameter $k_t = (I_a - I_t)/I_t$ and the spin parameter $Q = -\mu_2 I_a / I_t$. By inspection, the general solutions to this equation are[6]

$$\begin{array}{lll} \Lambda_3 = 0 & \widetilde{\Omega}_1 = \widetilde{\Omega}_2 = 0 & \text{(Cylindrical)} \\ \Lambda_3 = 0 & \widetilde{\Omega}_3 = Q/k_t & \text{(Hyperbolic)} \\ \Lambda_3 \neq 0 & & \text{(Conical).} \end{array}$$

For $\Lambda_3 = 0$, the gradient of the potential is aligned with the radius ($\nabla V_2(\boldsymbol{\Lambda}_e) = b_t \boldsymbol{\Lambda}_e$) and, by Equation (7.62b), we find $\widetilde{\boldsymbol{\Omega}}_e \cdot \boldsymbol{\Lambda}_e = 0$. Thus, the cylindrical and hyperbolic relative equilibria must have orthogonal orbits. Conversely, for conical relative equilibria with

[6] A fourth trivial solution exists for $k_t = Q = 0$ which we do not treat here. For a non-spinning axisymmetric body, we may revert back to the analysis for the arbitrary body to show that the orbit must be orthogonal and the symmetry axis must be along one of the orbital frame axes.

$\Lambda_3 \neq 0$, $\nabla V_2(\mathbf{\Lambda}_e)$ is not aligned with the radius and *the orbit must be oblique* (excluding the case where $\Lambda_1 = \Lambda_2 = 0$). The cylindrical and hyperbolic solutions are represented by configurations 8–10 of Figures 7.3 and 7.4 while the conical solutions represent configurations 1–7 and 11–17. As with the arbitrary body, the orthogonal relative equilibria give results which mirror the findings for the Keplerian system. The analysis of the conical solutions is complicated considerably by the absence of orthogonality and has not been completed. We present the analysis for the cylindrical relative equilibria which have practical application.

7.2.4 Cylindrical Relative Equilibria (2nd-Order Approximation).

Spectral Stability. The cylindrical solution to the Pringle-Likins Condition is $\Lambda_3 = 0$ and $\widetilde{\Omega}_1 = \widetilde{\Omega}_2 = 0$. The symmetry axis is normal to the orbit plane and we take the positive direction to be in the nodal frame angular velocity vector direction. Without loss of generality, we may choose the other two nodal frame basis vectors such that the first is directed in the radial direction and the second is in the tangential direction. Let $R = |\mathbf{\Lambda}_e|$ and $\omega = |\widetilde{\mathbf{\Omega}}_e|$. Then the relative equilibrium conditions of Equation (7.60) may be used to determine the full solution in terms of the original phase variables as

$$\begin{gathered} \mathbf{\Sigma}_e = (0, R\omega, 0) \qquad \mathbf{\Lambda}_e = (R, 0, 0) \qquad \mathbf{\Pi}_e = (0, 0, I_a(\omega + \mu_2)) \\ \mu_1^{-1} = \beta = I_a + R^2 - \frac{I_t Q}{\omega}. \end{gathered} \tag{7.72}$$

Furthermore, the generalization of Kepler's third law for the relationship between angular velocity and radius is

$$\omega^2 = \frac{3 - 9I_t + 2R^2}{2R^5}. \tag{7.73}$$

Inserting this solution into the linear system matrix of Equation (7.16), we find

$$\mathbf{A}(\mathbf{z}_e) =$$

$$\begin{bmatrix}
0 & \omega & 0 & \frac{-2(1-2b_tR^3)}{R^3} & 0 & 0 & 0 & 0 & \frac{(3+k_t)\omega R}{1+k_t} \\
-\omega & 0 & 0 & 0 & -b_t & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -b_a & -(3+k_t)\omega R & 0 & 0 \\
1 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -\omega & 0 & 0 & 0 & 0 & \frac{-(3+k_t)\omega R}{1+k_t} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & (3+k_t)\omega R & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & Q-k_t\omega & 0 \\
0 & 0 & 0 & 0 & 0 & (b_t-b_a)R & -Q+k_t\omega & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{7.74}$$

where we use the notation of the previous section for the gradient of the second-order potential ($\nabla V_2(\mathbf{\Lambda}_e) = (b_t\Lambda_1, b_t\Lambda_2, b_a\Lambda_3)$). This matrix decouples into a pitch subsystem associated with $(\Sigma_1, \Sigma_2, \Lambda_1, \Lambda_2, \Pi_3)$ and a roll-yaw subsystem associated with $(\Sigma_3, \Lambda_3, \Pi_1, \Pi_2)$. Because we have one Casimir function and one symmetry integral, we are assured the linear system matrix has three zero eigenvalues. In this case, they are all associated with the pitch subsystem and the characteristic equations are

$$P_p(s) = s^3(s^2 + A_0) \tag{7.75a}$$

$$P_{ry}(s) = s^4 + B_2 s^2 + B_0 \tag{7.75b}$$

where

$$A_0 = \frac{1}{2(3+k_t)R^5}(-3k_t + 6R^2 + 2k_tR^2) \tag{7.76a}$$

$$\begin{aligned} B_0 = {} & \frac{1}{4(3+k_t)^2R^{1}0}(3k_t + 6R^2 + 2k_tR^2)(k_t - S) \\ & \cdot (9k_t^2 + 24k_tR^2 + 8k_t^2R^2 - 9k_tS - 6R^2S - 2k_tR^2S) \end{aligned} \tag{7.76b}$$

$$B_2 = \frac{1}{2(3+k_t)R^5}(9k_t + 3k_t^3 + 6R^2 + 20k_t R^2 + 12k_t^2 R^2 + 2k_t^3 R^2$$
$$- 6k_t^2 S - 12k_t R^2 S - 4k_t^2 R^2 S + 3k_t S^2 + 6R^2 S^2 + 2k_t R^2 S^2). \tag{7.76c}$$

Note that in order to compensate for the different time scale used in the nondimensionalization, we have defined a new spin parameter $S = Q/\omega = -(I_a\mu_2)/(I_t\omega)$. The spectral stability conditions from Table 3.1 are

$$A_0 \geq 0 \qquad B_0 \geq 0 \qquad B_2 \geq 0 \qquad B_2^2 - 4B_0 \geq 0$$

and, upon eliminating strictly positive factors, may be expressed as polynomials in $1/R$:

$$2(3+k_t) - 3k_t\left(\frac{1}{R}\right)^2 \geq 0 \tag{7.77a}$$
$$(k_t - S)\left[(4k_t - S) + \frac{3k_t(7k_t - 4S)}{2(3+k_t)}\left(\frac{1}{R}\right)^2 + \frac{27k_t^2(k_t - S)}{4(3+k_t)^2}\left(\frac{1}{R}\right)^4\right] \geq 0 \tag{7.77b}$$
$$(1 + 3k_t + k_t^2 - 2k_t S + S^2) + \frac{3k_t(3 + k_t^2 - 2k_t S + S^2)}{2(3+k_t)}\left(\frac{1}{R}\right)^2 \geq 0 \tag{7.77c}$$
$$f_1 + \frac{3k_t}{3+k_t} f_2\left(\frac{1}{R}\right)^2 + \frac{9k_t^2(-3 + k_t^2 - 2k_t S + S^2)^2}{4(3+k_t)^2}\left(\frac{1}{R}\right)^4 \geq 0 \tag{7.77d}$$

where

$$f_1 = 1 + 6k_t - 5k_t^2 + 6k_t^3 + k_t^4 + 16k_t S - 12k_t^2 S - 4k_t^3 S$$
$$- 2S^2 + 6k_t S^2 + 6k_t^2 S^2 - 4k_t S^3 + S^4$$
$$f_2 = 3 + 9k_t - 10k_t^2 + 3k_t^3 + k_t^4 + 14k_t S - 6k_t^2 S - 4k_t^3 S$$
$$- 4S^2 + 3k_t S^2 + 6k_t^2 S^2 - 4k_t S^3 + S^4.$$

For all but very small values of R we may take just the zeroth-order terms, which are precisely the conditions we found for the Keplerian system (accounting for the difference in spin parameter). The resulting stability diagram was shown in Figure 6.4. As with the

arbitrary body, we may show that for certain configurations (prolate bodies), the radius has a lower bound.

Nonlinear Stability. For the Keplerian system, we found that cylindrical relative equilibria with configurations in the region given by the conditions $k_t - Q > 0$ and $4k_t - Q > 0$ were nonlinearly stable. We will now demonstrate the same result for the free rigid body when $k_t - S > 0$ and $4k_t - S > 0$. For the cylindrical relative equilibria, the Hessian of the variational Lagrangian given in Equation (7.15) becomes

$$\nabla^2 F(\mathbf{z}_e) =$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta - R^2}{\beta} & 0 & \frac{-(\beta+R^2)\omega}{\beta} & 0 & 0 & 0 & 0 & \frac{-R}{\beta} \\ 0 & 0 & \frac{\beta - R^2}{\beta} & 0 & 0 & 0 & 0 & \frac{R}{\beta} & 0 \\ 0 & \frac{-(\beta+R^2)\omega}{\beta} & 0 & -4b_t + \frac{2}{R^3} - \frac{\omega^2 R^2}{\beta} & 0 & 0 & 0 & 0 & \frac{-\omega R}{\beta} \\ \omega & 0 & 0 & 0 & b_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_a - \frac{\omega^2 R^2}{\beta} & \frac{\omega R}{\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\omega R}{\beta} & \frac{\beta - I_t}{\beta I_t} & 0 & 0 \\ 0 & 0 & \frac{R}{\beta} & 0 & 0 & 0 & 0 & \frac{\beta - I_t}{\beta I_t} & 0 \\ 0 & \frac{-R}{\beta} & 0 & \frac{-\omega R}{\beta} & 0 & 0 & 0 & 0 & \frac{\beta - I_a}{\beta I_a} \end{bmatrix}. \tag{7.78}$$

This matrix can be shown to be indefinite. We use the projection method to determine the conditions for a constrained extremum on the tangent space to the constraint manifold. Because the system has symmetry, we also restrict the definition of stability to omit the cyclic perturbations given by

$$\boldsymbol{\xi}(\mathbf{z}_e) = (\mathbf{1}_3{}^{\times}\boldsymbol{\Sigma}_e, \mathbf{1}_3{}^{\times}\boldsymbol{\Lambda}_e, \mathbf{1}_3{}^{\times}\boldsymbol{\Pi}_e) = (-\omega R, 0, 0, 0, R, 0, 0, 0, 0). \tag{7.79}$$

The resulting projection $\mathbf{P}(\mathbf{z}_e) = \mathbf{1} - \mathbf{K}(\mathbf{z}_e)\left[\mathbf{K}^{\mathsf{T}}(\mathbf{z}_e)\mathbf{K}(\mathbf{z}_e)\right]^{-1}\mathbf{K}^{\mathsf{T}}(\mathbf{z}_e)$ with

$$\mathbf{K}(\mathbf{z}_e) = \begin{bmatrix} \nabla C_1(\mathbf{z}_e) & \nabla C_2(\mathbf{z}_e) & \boldsymbol{\xi}(\mathbf{z}_e) \end{bmatrix} \tag{7.80}$$

has rank six and we look for conditions for definiteness of the Hessian on this constrained space.

The Hessian of the variational Lagrangian can be block-diagonalized by reordering the phase variables as $(\Sigma_1, \Lambda_2, \Sigma_2, \Lambda_1, \Pi_3, \Sigma_3, \Pi_2, \Lambda_3, \Pi_1)$. The diagonal would then consist of a 2×2 block associated with Σ_1 and Λ_2, a 3×3 block associated with Σ_2, Λ_2, and Π_3, a 2×2 block associated with Σ_3 and Π_2, and a 2×2 block associated with Λ_3 and Π_1. The projection only affects the subspaces associated with the first and second blocks. The projected Hessian (with the original order of variables) is

$$\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e) =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & \frac{b_t(2-b_tR^3)}{(1+b_t)^2R^3} & 0 & \frac{-\omega(2-b_tR^3)}{(1+b_t)^2R^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta-R^2}{\beta} & 0 & 0 & 0 & 0 & \frac{R}{\beta} & 0 \\ 0 & \frac{-\omega(2-b_tR^3)}{(1+b_t)^2R^3} & 0 & \frac{2-b_tR^3}{(1+b_t)^2R^3} & 0 & 0 & 0 & 0 & 0 \\ \omega & 0 & 0 & 0 & b_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_a - \frac{\omega^2R^2}{\beta} & \frac{\omega R}{\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\omega R}{\beta} & \frac{\beta - I_t}{\beta I_t} & 0 & 0 \\ 0 & 0 & \frac{R}{\beta} & 0 & 0 & 0 & 0 & \frac{\beta-I_t}{\beta I_t} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (7.81)$$

In preference to determining the eigenvalues, we again compute the pivots for each of the blocks using LDL decomposition:

$$\begin{aligned}
&(\Sigma_1, \Lambda_2) && \{1, 0\} \\
&(\Sigma_2, \Lambda_1, \Pi_3) && \left\{\frac{(-3k_t + 6R^2 + 2k_tR^2)(3k_t + 6R^2 + 2k_tR^2)}{(3k_t + 6R^2 + 2k_tR^2 + 6R^5 + 2k_tR^5)^2}, 0, 0\right\} \\
&(\Sigma_3, \Pi_2) && \left\{\frac{1 + k_t - S}{1 + k_t + 3R^2 + k_tR^2 - S}, \frac{(3 + k_t)(k_t - S)}{1 + k_t - S}\right\} \\
&(\Lambda_3, \Pi_1) && \left\{\frac{9k_t + 9k_t^2 + 6R^2 + 26k_tR^2 + 8k_t^2R^2 - 9k_tS - 6R^2S - 2k_tR^2S}{2(3 + k_t)R^5(1 + k_t + 3R^2 + k_tR^2 - S)}, \right. \\
& && \left. \frac{(3 + k_t)(9k_t^2 + 24k_tR^2 + 8k_t^2R^2 - 9k_tS - 6R^2S - 2k_tR^2S)}{9k_t + 9k_t^2 + 6R^2 + 26k_tR^2 + 8k_t^2R^2 - 9k_tS - 6R^2S - 2k_tR^2S}\right\}.
\end{aligned}$$

The three zero pivots are associated with the Casimir and symmetry integrals and the cyclic perturbation direction. The remaining six pivots are of the same sign as the eigenvalues on the restricted tangent space. The first pivot in the first block assures our results are restricted to the positive definite case. The conditions for positive definiteness may be expressed as

$$4(3+k_t)^2 - 9k_t^2\left(\frac{1}{R}\right)^4 > 0 \qquad (7.82a)$$

$$(1+k_t-S)\left[(3+k_t)+(1+k_t-S)\left(\frac{1}{R}\right)^2\right] > 0 \qquad (7.82b)$$

$$(k_t-S)(1+k_t-S) > 0 \qquad (7.82c)$$

$$2(3+k_t)^2(1+4k_t-S)+(3+k_t)(2+17k_t-2S)(1+k_t-S)\left(\frac{1}{R}\right)^2 + 9k_t(1+k_t-S)^2\left(\frac{1}{R}\right)^4 > 0 \qquad (7.82d)$$

$$4(3+k_t)^2(4k_t-S)(1+4k_t-S) + 18k_t(3+k_t)(5k_t+8k_t^2-2S-10k_tS+2S^2)\left(\frac{1}{R}\right)^2 + 81k_t^2(k_t-S)(1+k_t-S)\left(\frac{1}{R}\right)^4 > 0 \qquad (7.82e)$$

which, for sufficiently large R, reduces to $k_t - S > 0$ and $4k_t - S > 0$. In contrast to the arbitrary body, we cannot determine a lower bound for which this region is nonlinearly stable. Analysis of the second pivot in the (Λ_3, Π_1) block demonstrates that, for $k_t < 0$, the boundary given by $4k_t - S = 0$ is an asymptotic limit as $R \to \infty$. However, for all practical values of R, the true boundary must be very close to this asymptotic limit. Thus, the results for the Keplerian system have been extended to the free rigid-body system with a second-order potential approximation.

7.3 Conclusions

We have extended much of the results of the Keplerian system to the free rigid-body system via Hamiltonian methods. While some of the results for the free rigid body depend

on the distance from the center of attraction, in general the results are identical to those of the Keplerian system for all but very small values of $|\mathbf{\Lambda}_e|$. The one unique finding of note is the presence of a class of relative equilibria in the axisymmetric case for which the orbits are oblique. We next look at how some of the classical results may also be extended to the free rigid-body system with the exact potential and investigate some results which are unique to higher order approximations.

VIII. *Free Rigid Body System: Exact Potential*

In the preceding chapter, we developed the general relative equilibrium conditions for a free rigid body in a central gravitational field. We then assumed a second-order approximation of the potential and demonstrated that the (classical) Keplerian solutions can be extended to the free rigid body where the dynamics of the orbit and attitude are coupled. We considered both the arbitrary and axisymmetric cases and found that only the conical relative equilibria of the axisymmetric body allow (and require) oblique orbits. Certain symmetries are implicit in the second-order assumption which lead to these results.

For the untruncated potential, Wang *et al* [104, 105] have shown the possibility of oblique relative equilibria. Moreover, they have proven the existence of oblique relative equilibria for a specific rigid body. They also developed sufficient conditions for the existence of orthogonal relative equilibria in terms of symmetries present in a rigid body. These criteria were used to determine orthogonal solutions for an example body and numerical continuation methods were then applied to locate oblique relative equilibria. They demonstrated that, while the displacement of oblique orbits is very small, the associated variation in attitude can be significant for certain inertia configurations. This work serves as the basis for the current effort.

In this chapter, we focus on the presence of orthogonal relative equilibria for the exact potential. The general developments of Chapter 7 serve as a starting point to define conditions for their existence and examine linear and nonlinear stability of the orbits. We pay particular attention to bodies with one or more planes of symmetry for which the existence of orthogonal relative equilibria is guaranteed. The axisymmetric case is not considered. Analysis of an example rigid body is presented to clarify our discussion.

8.1 Orthogonal Relative Equilibria

8.1.1 Requirements for Orthogonality. In Section 7.1.2, we found the conditions for relative equilibrium by determining the critical points of the variational Lagrangian. These

conditions were reduced to the form

$$\nabla V(\mathbf{\Lambda}_e) + \mathbf{\Omega}_e^\times \mathbf{\Omega}_e^\times \mathbf{\Lambda}_e = \mathbf{0} \tag{7.12a}$$

$$\beta \mathbf{\Omega}_e - \mathbf{I}\mathbf{\Omega}_e + \mathbf{\Lambda}_e^\times \mathbf{\Lambda}_e^\times \mathbf{\Omega}_e = \mathbf{0} \tag{7.12b}$$

$$c - \frac{1}{2}\mathbf{\Omega}_e \cdot \mathbf{\Omega}_e = 0. \tag{7.12c}$$

We now restrict our consideration to orthogonal orbits with the defining condition $\mathbf{\Omega}_e \cdot \mathbf{\Lambda}_e \equiv 0$. The relative equilibrium conditions in Equation (7.12) then reduce to

$$\nabla V(\mathbf{\Lambda}_e) - |\mathbf{\Omega}_e|^2 \mathbf{\Lambda}_e = \mathbf{0} \tag{8.2a}$$

$$\beta \mathbf{\Omega}_e - \mathbf{I}\mathbf{\Omega}_e - |\mathbf{\Lambda}_e|^2 \mathbf{\Omega}_e = \mathbf{0} \tag{8.2b}$$

$$c - \frac{1}{2}|\mathbf{\Omega}_e|^2 = 0 \tag{8.2c}$$

These are the orthogonal relative equilibrium conditions. Equation (8.2a) requires that the total force on the body act in the radial direction. Equation (8.2b), which may be written as

$$\mathbf{I}\mathbf{\Omega}_e = (\beta - |\mathbf{\Lambda}_e|^2)\mathbf{\Omega}_e, \tag{8.2b$'$}$$

requires that the angular velocity be directed along a principal inertia axis of the rigid body. By Remark 3 of Section 7.1.2, these findings are unique to orthogonal relative equilibria.

In the previous analysis of Chapters 6 and 7, we specified the body frame as a principal frame and then sought solutions for the vectors $\mathbf{\Omega}_e$ and $\mathbf{\Lambda}_e$. In this chapter, we take the opposite approach and specify a body frame aligned with the orbital frame as shown in Figure 8.1. (We do not require this frame to be a principal axis frame as in the classical solution.) We then seek conditions on the inertias and potential in terms of the radius and angular velocity. Let $|\mathbf{\Lambda}_e| = R$ and $|\mathbf{\Omega}_e| = \omega$. For this frame, at relative equilibrium we

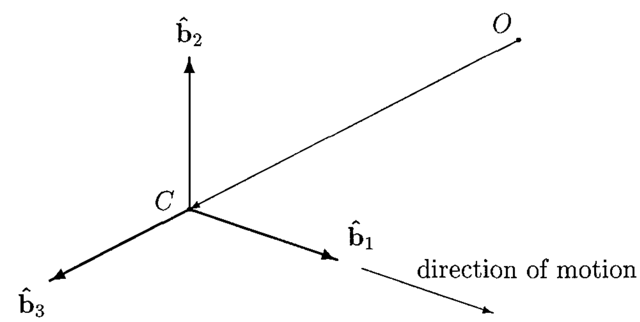


Figure 8.1 Body Frame at Relative Equilibrium

have $\mathbf{\Lambda}_e = (0, 0, R)$ and $\mathbf{\Omega}_e = (0, \omega, 0)$ and Equation (8.2) gives

$$v_1 = 0 \qquad I_{12}\omega = 0 \tag{8.3a}$$

$$v_2 = 0 \qquad (I_{22} + R^2 - \beta)\omega = 0 \tag{8.3b}$$

$$\omega^2 R + v_3 = 0 \qquad I_{23}\omega = 0 \tag{8.3c}$$

where we have set $\nabla V(\mathbf{\Lambda}_e) = (v_1, v_2, v_3)$ and

$$\mathbf{I} = \begin{bmatrix} I_{11} & -I_{12} & -I_{13} \\ -I_{12} & I_{22} & -I_{23} \\ -I_{13} & -I_{23} & I_{33} \end{bmatrix}.$$

Since $\mathbf{\Omega}_e$ is an eigenvector of the inertia matrix, we find that $I_{12} = I_{23} = 0$ is required for equilibrium. We set $I_{22} = I_2$ to indicate it is a principal moment of inertia. We also conclude that $\beta = I_2 + R^2$.

While the conditions given in Equation (8.3) appear relatively simple, the components of the gradient of the potential (v_i, $i = 1, 2, 3$) are integrals over the body and are typically not solvable in closed form. Without expanding the integrand in a series and truncating, there is little we can say about the implications of these conditions. Furthermore, we are limited in our ability to make general conclusions regarding stability, as we will discover momentarily. It would appear that the conditions on the gradient would have some result-

ing effect on the Hessian, but that effect has not been determined. This is suggested as a topic for future research.

8.1.2 Spectral Stability. The system matrix for the linearization of the equations of motion was given in Section 7.1.2 as

$$\mathbf{A}(\mathbf{z}_e) = \begin{bmatrix} -\boldsymbol{\Omega}_e^\times & -\nabla^2 V(\boldsymbol{\Lambda}_e) & \boldsymbol{\Sigma}_e^\times \mathbf{I}^{-1} \\ \mathbf{1} & -\boldsymbol{\Omega}_e^\times & \boldsymbol{\Lambda}_e^\times \mathbf{I}^{-1} \\ \mathbf{0} & \boldsymbol{\Lambda}_e^\times \nabla^2 V(\boldsymbol{\Lambda}_e) + \left(\boldsymbol{\Omega}_e^\times \boldsymbol{\Sigma}_e\right)^\times & (\mathbf{I}\boldsymbol{\Omega}_e)^\times \mathbf{I}^{-1} - \boldsymbol{\Omega}_e^\times \end{bmatrix}. \tag{7.16}$$

At an orthogonal relative equilibrium in our chosen body frame, this is

$$\mathbf{A}(\mathbf{z}_e) =$$
$$\begin{bmatrix} 0 & 0 & -\omega & -v_{11} & -v_{12} & -v_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & -v_{12} & -v_{22} & -v_{23} & \frac{-I_{13}\omega R}{\Delta} & 0 & \frac{-I_{11}\omega R}{\Delta} \\ \omega & 0 & 0 & -v_{13} & -v_{23} & -v_{33} & 0 & \frac{\omega R}{I_2} & 0 \\ 1 & 0 & 0 & 0 & 0 & -\omega & 0 & \frac{-R}{I_2} & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{I_{33}R}{\Delta} & 0 & \frac{I_{13}R}{\Delta} \\ 0 & 0 & 1 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -Rv_{12} & -R(v_{22}-\omega^2) & -Rv_{23} & \frac{I_2 I_{13}\omega}{\Delta} & 0 & \omega(\frac{I_2 I_{11}}{\Delta}-1) \\ 0 & 0 & 0 & R(v_{11}-\omega^2) & Rv_{12} & Rv_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\omega(\frac{I_2 I_{33}}{\Delta}-1) & 0 & \frac{-I_2 I_{13}\omega}{\Delta} \end{bmatrix} \tag{8.4}$$

where $\Delta = I_{11}I_{33} - I_{13}^2$. We have a single Casimir function and no symmetry integrals, so the characteristic polynomial for $\mathbf{A}(\mathbf{z}_e)$ takes the form

$$P(s) = s(s^8 + A_6 s^6 + A_4 s^4 + A_2 s^2 + A_0). \tag{8.5}$$

Examination of the system matrix reveals that the linear system is completely coupled unless $v_{12} = v_{23} = 0$ in which case the system decouples into a pitch system associated with $(\Sigma_1, \Sigma_3, \Lambda_1, \Lambda_3, \Pi_2)$ and a roll-yaw subsystem associated with $(\Sigma_2, \Lambda_2, \Pi_1, \Pi_3)$. This

decoupling occurs for the systems with symmetry we will consider below. For the general coupled case, the coefficients of this polynomial are complicated functions of the potential terms, the inertias, and the radius. They are given in Table 8.1. The stability conditions for this polynomial are given in Table 3.1. For a given rigid body at a known relative equilibrium, we may compute the coefficients. However, at this point we can draw no general conclusions regarding spectral stability.

Table 8.1: Coefficients of the Characteristic Equation for Orthogonal Relative Equilibria

$$
\begin{aligned}
A_0 = \quad & \tfrac{-v_3}{I_2 R^3 \Delta}(-I_{11} I_2^2 R^2 v_{13}^2 v_{22} + I_2^3 R^2 v_{13}^2 v_{22} - I_2^2 I_{33} R^2 v_{13}^2 v_{22} - 2 I_{11} I_2 R^4 v_{13}^2 v_{22} + 2 I_2^2 R^4 v_{13}^2 v_{22} \\
& - I_2 I_{33} R^4 v_{13}^2 v_{22} - I_{11} R^6 v_{13}^2 v_{22} + I_2 R^6 v_{13}^2 v_{22} + 2 I_{11} I_2^2 R^2 v_{12} v_{13} v_{23} - 2 I_2^3 R^2 v_{12} v_{13} v_{23} \\
& + 2 I_2^2 I_{33} R^2 v_{12} v_{13} v_{23} + 4 I_{11} I_2 R^4 v_{12} v_{13} v_{23} - 4 I_2^2 R^4 v_{12} v_{13} v_{23} + 2 I_2 I_{33} R^4 v_{12} v_{13} v_{23} \\
& + 2 I_{11} R^6 v_{12} v_{13} v_{23} - 2 I_2 R^6 v_{12} v_{13} v_{23} - I_{11} I_2^2 R^2 v_{11} v_{23}^2 + I_2^3 R^2 v_{11} v_{23}^2 \\
& - I_2^2 I_{33} R^2 v_{11} v_{23}^2 - 2 I_{11} I_2 R^4 v_{11} v_{23}^2 + 2 I_2^2 R^4 v_{11} v_{23}^2 - I_2 I_{33} R^4 v_{11} v_{23}^2 - I_{11} R^6 v_{11} v_{23}^2 \\
& + I_2 R^6 v_{11} v_{23}^2 + I_{11} I_2^2 R v_{12}^2 v_3 - I_2^3 R v_{12}^2 v_3 + I_2^2 I_{33} R v_{12}^2 v_3 - 2 I_{11} I_2 R^3 v_{12}^2 v_3 \\
& + 2 I_2^2 R^3 v_{12}^2 v_3 - 3 I_2 I_{33} R^3 v_{12}^2 v_3 - 3 I_{11} R^5 v_{12}^2 v_3 + 3 I_2 R^5 v_{12}^2 v_3 + I_{11} I_2 R^3 v_{13}^2 v_3 \\
& - I_2^2 R^3 v_{13}^2 v_3 + I_{11} R^5 v_{13}^2 v_3 - I_2 R^5 v_{13}^2 v_3 - I_{11} I_2^2 R v_{11} v_{22} v_3 + I_2^3 R v_{11} v_{22} v_3 \\
& - I_2^2 I_{33} R v_{11} v_{22} v_3 + 2 I_{11} I_2 R^3 v_{11} v_{22} v_3 - 2 I_2^2 R^3 v_{11} v_{22} v_3 + 3 I_2 I_{33} R^3 v_{11} v_{22} v_3 \\
& + 3 I_{11} R^5 v_{11} v_{22} v_3 - 3 I_2 R^5 v_{11} v_{22} v_3 + I_{11} I_2^2 R v_{23}^2 v_3 - I_2^3 R v_{23}^2 v_3 + I_2^2 I_{33} R v_{23}^2 v_3 \\
& + 2 I_{11} I_2 R^3 v_{23}^2 v_3 - 2 I_2^2 R^3 v_{23}^2 v_3 + I_2 I_{33} R^3 v_{23}^2 v_3 + I_{11} R^5 v_{23}^2 v_3 - I_2 R^5 v_{23}^2 v_3 \\
& + I_{11} I_2 R^2 v_{11} v_3^2 - I_2^2 R^2 v_{11} v_3^2 - 3 I_{11} R^4 v_{11} v_3^2 + 3 I_2 R^4 v_{11} v_3^2 + I_{11} I_2^2 v_{22} v_3^2 - I_2^3 v_{22} v_3^2 \\
& + I_2^2 I_{33} v_{22} v_3^2 - 2 I_{11} I_2 R^2 v_{22} v_3^2 + 2 I_2^2 R^2 v_{22} v_3^2 - 3 I_2 I_{33} R^2 v_{22} v_3^2 - 3 I_{11} R^4 v_{22} v_3^2 \\
& + 3 I_2 R^4 v_{22} v_3^2 - I_{11} I_2 R v_3^3 + I_2^2 R v_3^3 + 3 I_{11} R^3 v_3^3 - 3 I_2 R^3 v_3^3 - I_{11} I_2^2 R^2 v_{12}^2 v_{33} \\
& + I_2^3 R^2 v_{12}^2 v_{33} - I_2^2 I_{33} R^2 v_{12}^2 v_{33} - 2 I_{11} I_2 R^4 v_{12}^2 v_{33} + 2 I_2^2 R^4 v_{12}^2 v_{33} - I_2 I_{33} R^4 v_{12}^2 v_{33} \\
& - I_{11} R^6 v_{12}^2 v_{33} + I_2 R^6 v_{12}^2 v_{33} + I_{11} I_2^2 R^2 v_{11} v_{22} v_{33} - I_2^3 R^2 v_{11} v_{22} v_{33} + I_2^2 I_{33} R^2 v_{11} v_{22} v_{33} \\
& + 2 I_{11} I_2 R^4 v_{11} v_{22} v_{33} - 2 I_2^2 R^4 v_{11} v_{22} v_{33} + I_2 I_{33} R^4 v_{11} v_{22} v_{33} + I_{11} R^6 v_{11} v_{22} v_{33} \\
& - I_2 R^6 v_{11} v_{22} v_{33} - I_{11} I_2 R^3 v_{11} v_3 v_{33} + I_2^2 R^3 v_{11} v_3 v_{33} - I_{11} R^5 v_{11} v_3 v_{33} + I_2 R^5 v_{11} v_3 v_{33} \\
& - I_{11} I_2^2 R v_{22} v_3 v_{33} + I_2^3 R v_{22} v_3 v_{33} - I_2^2 I_{33} R v_{22} v_3 v_{33} - 2 I_{11} I_2 R^3 v_{22} v_3 v_{33} \\
& + 2 I_2^2 R^3 v_{22} v_3 v_{33} - I_2 I_{33} R^3 v_{22} v_3 v_{33} - I_{11} R^5 v_{22} v_3 v_{33} + I_2 R^5 v_{22} v_3 v_{33} + I_{11} I_2 R^2 v_3^2 v_{33} \\
& - I_2^2 R^2 v_3^2 v_{33} + I_{11} R^4 v_3^2 v_{33} - I_2 R^4 v_3^2 v_{33} + I_2 R^2 v_{13}^2 v_{22} \Delta + R^4 v_{13}^2 v_{22} \Delta \\
& - 2 I_2 R^2 v_{12} v_{13} v_{23} \Delta - 2 R^4 v_{12} v_{13} v_{23} \Delta + I_2 R^2 v_{11} v_{23}^2 \Delta + R^4 v_{11} v_{23}^2 \Delta - I_2 R v_{12}^2 v_3 \Delta \\
& + 3 R^3 v_{12}^2 v_3 \Delta + I_2 R v_{11} v_{22} v_3 \Delta - 3 R^3 v_{11} v_{22} v_3 \Delta - I_2 R v_{23}^2 v_3 \Delta - R^3 v_{23}^2 v_3 \Delta \\
& - I_2 v_{22} v_3^2 \Delta + 3 R^2 v_{22} v_3^2 \Delta + I_2 R^2 v_{12}^2 v_{33} \Delta + R^4 v_{12}^2 v_{33} \Delta - I_2 R^2 v_{11} v_{22} v_{33} \Delta \\
& - R^4 v_{11} v_{22} v_{33} \Delta + I_2 R v_{22} v_3 v_{33} \Delta + R^3 v_{22} v_3 v_{33} \Delta)
\end{aligned}
$$

Table 8.1: Coefficients of the Characteristic Equation for Orthogonal Relative Equilibria (cont.)

$$
\begin{aligned}
A_2 = \; & \tfrac{-1}{I_2 R^3 \Delta}(I_2 I_{33} R^5 v_{13}^2 v_{22} + I_{33} R^7 v_{13}^2 v_{22} - 2 I_2 I_{33} R^5 v_{12} v_{13} v_{23} - 2 I_{33} R^7 v_{12} v_{13} v_{23} \\
& + I_2 I_{33} R^5 v_{11} v_{23}^2 + I_{33} R^7 v_{11} v_{23}^2 - I_{11} I_2^2 R^2 v_{12}^2 v_3 + I_2^3 R^2 v_{12}^2 v_3 - I_2^2 I_{33} R^2 v_{12}^2 v_3 \\
& - 2 I_{11} I_2 R^4 v_{12}^2 v_3 + 2 I_2^2 R^4 v_{12}^2 v_3 - 2 I_2 I_{33} R^4 v_{12}^2 v_3 - I_{11} R^6 v_{12}^2 v_3 + I_2 R^6 v_{12}^2 v_3 \\
& + 3 I_{33} R^6 v_{12}^2 v_3 - I_{11} I_2^2 R^2 v_{13}^2 v_3 + I_2^3 R^2 v_{13}^2 v_3 - I_2^2 I_{33} R^2 v_{13}^2 v_3 - I_{11} I_2 R^4 v_{13}^2 v_3 \\
& + I_2^2 R^4 v_{13}^2 v_3 - 2 I_2 I_{33} R^4 v_{13}^2 v_3 - I_{33} R^6 v_{13}^2 v_3 + I_{11} I_2^2 R^2 v_{11} v_{22} v_3 - I_2^3 R^2 v_{11} v_{22} v_3 \\
& + I_2^2 I_{33} R^2 v_{11} v_{22} v_3 + 2 I_{11} I_2 R^4 v_{11} v_{22} v_3 - 2 I_2^2 R^4 v_{11} v_{22} v_3 + 2 I_2 I_{33} R^4 v_{11} v_{22} v_3 \\
& + I_{11} R^6 v_{11} v_{22} v_3 - I_2 R^6 v_{11} v_{22} v_3 - 3 I_{33} R^6 v_{11} v_{22} v_3 - I_{11} I_2^2 R^2 v_{23}^2 v_3 + I_2^3 R^2 v_{23}^2 v_3 \\
& - I_2^2 I_{33} R^2 v_{23}^2 v_3 - I_{11} I_2 R^4 v_{23}^2 v_3 + I_2^2 R^4 v_{23}^2 v_3 - I_2 I_{33} R^4 v_{23}^2 v_3 - I_{33} R^6 v_{23}^2 v_3 \\
& - I_{11} I_2^2 R v_{11} v_3^2 + I_2^3 R v_{11} v_3^2 - I_2^2 I_{33} R v_{11} v_3^2 + 2 I_{11} I_2 R^3 v_{11} v_3^2 - 2 I_2^2 R^3 v_{11} v_3^2 \\
& + 2 I_2 I_{33} R^3 v_{11} v_3^2 - I_{11} R^5 v_{11} v_3^2 + I_2 R^5 v_{11} v_3^2 + 3 I_{33} R^5 v_{11} v_3^2 + 2 I_{11} I_2^2 R v_{22} v_3^2 \\
& - 2 I_2^3 R v_{22} v_3^2 + 2 I_2^2 I_{33} R v_{22} v_3^2 + I_{11} I_2 R^3 v_{22} v_3^2 - I_2^2 R^3 v_{22} v_3^2 - 2 I_2 I_{33} R^3 v_{22} v_3^2 \\
& - I_{11} R^5 v_{22} v_3^2 + I_2 R^5 v_{22} v_3^2 + 3 I_{33} R^5 v_{22} v_3^2 + I_{11} I_2^2 v_3^3 - I_2^3 v_3^3 + I_2^2 I_{33} v_3^3 - 5 I_{11} I_2 R^2 v_3^3 \\
& + 5 I_2^2 R^2 v_3^3 - 2 I_2 I_{33} R^2 v_3^3 + I_{11} R^4 v_3^3 - I_2 R^4 v_3^3 - 3 I_{33} R^4 v_3^3 + I_2 I_{33} R^5 v_{12}^2 v_{33} \\
& + I_{33} R^7 v_{12}^2 v_{33} - I_2 I_{33} R^5 v_{11} v_{22} v_{33} - I_{33} R^7 v_{11} v_{22} v_{33} + I_{11} I_2^2 R^2 v_{11} v_3 v_{33} \\
& - I_2^3 R^2 v_{11} v_3 v_{33} + I_2^2 I_{33} R^2 v_{11} v_3 v_{33} + I_{11} I_2 R^4 v_{11} v_3 v_{33} - I_2^2 R^4 v_{11} v_3 v_{33} \\
& + 2 I_2 I_{33} R^4 v_{11} v_3 v_{33} + I_{33} R^6 v_{11} v_3 v_{33} + I_{11} I_2^2 R^2 v_{22} v_3 v_{33} - I_2^3 R^2 v_{22} v_3 v_{33} \\
& + I_2^2 I_{33} R^2 v_{22} v_3 v_{33} + I_{11} I_2 R^4 v_{22} v_3 v_{33} - I_2^2 R^4 v_{22} v_3 v_{33} + I_2 I_{33} R^4 v_{22} v_3 v_{33} \\
& + I_{33} R^6 v_{22} v_3 v_{33} - I_{11} I_2^2 R v_3^2 v_{33} + I_2^3 R v_3^2 v_{33} - I_2^2 I_{33} R v_3^2 v_{33} - 2 I_{11} I_2 R^3 v_3^2 v_{33} \\
& + 2 I_2^2 R^3 v_3^2 v_{33} - 2 I_2 I_{33} R^3 v_3^2 v_{33} - I_{33} R^5 v_3^2 v_{33} + I_2 R^3 v_{13}^2 v_{22} \Delta + R^5 v_{13}^2 v_{22} \Delta \\
& - 2 I_2 R^3 v_{12} v_{13} v_{23} \Delta - 2 R^5 v_{12} v_{13} v_{23} \Delta + I_2 R^3 v_{11} v_{23}^2 \Delta + R^5 v_{11} v_{23}^2 \Delta + 4 R^4 v_{12}^2 v_3 \Delta \\
& + I_2 R^2 v_{13}^2 v_3 \Delta + R^4 v_{13}^2 v_3 \Delta - 4 R^4 v_{11} v_{22} v_3 \Delta - R^4 v_{23}^2 v_3 \Delta + I_2 R v_{11} v_3^2 \Delta \\
& - 3 R^3 v_{11} v_3^2 \Delta - 3 I_2 R v_{22} v_3^2 \Delta + 4 R^3 v_{22} v_3^2 \Delta - I_2 v_3^3 \Delta + 3 R^2 v_3^3 \Delta + I_2 R^3 v_{12}^2 v_{33} \Delta \\
& + R^5 v_{12}^2 v_{33} \Delta - I_2 R^3 v_{11} v_{22} v_{33} \Delta - R^5 v_{11} v_{22} v_{33} \Delta - I_2 R^2 v_{11} v_3 v_{33} \Delta - R^4 v_{11} v_3 v_{33} \Delta \\
& + R^4 v_{22} v_3 v_{33} \Delta + I_2 R v_3^2 v_{33} \Delta + R^3 v_3^2 v_{33} \Delta) \\
A_4 = \; & \tfrac{-1}{I_2 R^2 \Delta}(I_2 I_{33} R^4 v_{12}^2 + I_{33} R^6 v_{12}^2 - I_2 I_{33} R^4 v_{11} v_{22} - I_{33} R^6 v_{11} v_{22} + I_2 I_{33} R^4 v_{23}^2 \\
& + I_{11} I_2^2 R v_{11} v_3 - I_2^3 R v_{11} v_3 + I_2^2 I_{33} R v_{11} v_3 + I_{11} I_2 R^3 v_{11} v_3 - I_2^2 R^3 v_{11} v_3 + 2 I_2 I_{33} R^3 v_{11} v_3 \\
& + I_{33} R^5 v_{11} v_3 + I_{11} I_2^2 R v_{22} v_3 - I_2^3 R v_{22} v_3 + I_2^2 I_{33} R v_{22} v_3 + I_{11} I_2 R^3 v_{22} v_3 - I_2^2 R^3 v_{22} v_3 \\
& - 2 I_2 I_{33} R^3 v_{22} v_3 + I_{33} R^5 v_{22} v_3 + 2 I_{11} I_2^2 v_3^2 - 2 I_2^3 v_3^2 + 2 I_2^2 I_{33} v_3^2 - 2 I_{11} I_2 R^2 v_3^2 \\
& + 2 I_2^2 R^2 v_3^2 + I_2 I_{33} R^2 v_3^2 - I_{33} R^4 v_3^2 - I_2 I_{33} R^4 v_{22} v_{33} + I_{11} I_2^2 R v_3 v_{33} - I_2^3 R v_3 v_{33} \\
& + I_2^2 I_{33} R v_3 v_{33} + I_2 I_{33} R^3 v_3 v_{33} + I_2 R^2 v_{12}^2 \Delta + R^4 v_{12}^2 \Delta + I_2 R^2 v_{13}^2 \Delta + R^4 v_{13}^2 \Delta \\
& - I_2 R^2 v_{11} v_{22} \Delta - R^4 v_{11} v_{22} \Delta + I_2 R^2 v_{23}^2 \Delta - 4 R^3 v_{11} v_3 \Delta - 3 I_2 R v_{22} v_3 \Delta + R^3 v_{22} v_3 \Delta
\end{aligned}
$$

Table 8.1: Coefficients of the Characteristic Equation for Orthogonal Relative Equilibria (cont.)

	$-3I_2v_3^2\Delta + 4R^2v_3^2\Delta - I_2R^2v_{11}v_{33}\Delta - R^4v_{11}v_{33}\Delta - I_2R^2v_{22}v_{33}\Delta + R^3v_3v_{33}\Delta)$
$A_6 =$	$\frac{1}{I_2R\Delta}(I_2I_{33}R^3v_{22} - I_{11}I_2^2v_3 + I_2^3v_3 - I_2^2I_{33}v_3 - I_2I_{33}R^2v_3 + I_2Rv_{11}\Delta + R^3v_{11}\Delta$
	$+ I_2Rv_{22}\Delta + 3I_2v_3\Delta - R^2v_3\Delta + I_2Rv_{33}\Delta)$

8.1.3 Nonlinear Stability. We next apply the projection method to the orthogonal relative equilibria of the exact system. The Hessian of $F(\mathbf{z})$ for the free rigid-body system was given in Chapter 7 as

$$\nabla^2F(\mathbf{z}) = \begin{bmatrix} \mathbf{1} + \mu_1\mathbf{\Lambda}^\times\mathbf{\Lambda}^\times & -\mu_1\mathbf{\Gamma}^\times - \mu_1\mathbf{\Lambda}^\times\mathbf{\Sigma}^\times & \mu_1\mathbf{\Lambda}^\times \\ \mu_1\mathbf{\Gamma}^\times - \mu_1\mathbf{\Sigma}^\times\mathbf{\Lambda}^\times & \nabla^2V(\mathbf{\Lambda}) + \mu_1\mathbf{\Sigma}^\times\mathbf{\Sigma}^\times & -\mu_1\mathbf{\Sigma}^\times \\ -\mu_1\mathbf{\Lambda}^\times & \mu_1\mathbf{\Sigma}^\times & \mathbf{I}^{-1} - \mu_1\mathbf{1} \end{bmatrix} \tag{7.15}$$

To simplify matters somewhat, assume $\mathbf{z}_e$ is a principal relative equilibrium. Then in the body frame

$$\nabla^2F(\mathbf{z}_e) = \begin{bmatrix} \frac{I_2}{\beta} & 0 & 0 & 0 & 0 & \frac{-(\beta+R^2)}{\omega} & 0 & \frac{-R}{\beta} & 0 \\ 0 & \frac{I_2}{\beta} & 0 & 0 & 0 & 0 & \frac{R}{\beta} & 0 & 0 \\ 0 & 0 & 1 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & v_{11} & v_{12} & v_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & v_{12} & v_{22} - \frac{\omega^2R^2}{\beta} & v_{23} & 0 & 0 & \frac{\omega R}{\beta} \\ \frac{-(\beta+R^2)}{\omega} & 0 & 0 & v_{13} & v_{23} & v_{33} - \frac{\omega^2R^2}{\beta} & 0 & \frac{-\omega R}{\beta} & 0 \\ 0 & \frac{R}{\beta} & 0 & 0 & 0 & 0 & \frac{1}{I_1} - \frac{1}{\beta} & 0 & 0 \\ \frac{-R}{\beta} & 0 & 0 & 0 & 0 & \frac{-\omega R}{\beta} & 0 & \frac{1}{I_2} - \frac{1}{\beta} & 0 \\ 0 & 0 & 0 & 0 & \frac{\omega R}{\beta} & 0 & 0 & 0 & \frac{1}{I_3} - \frac{1}{\beta} \end{bmatrix} \tag{8.6}$$

which may be shown to be indefinite. We construct the projection onto the tangent space of the constraint manifold and compute the projected Hessian. This matrix is complicated due to the full coupling and analysis of the stability conditions is not pursued.

8.2 Rigid Bodies with Planes of Symmetry

In this section, we discuss the class of bodies with one or more planes of symmetry. For such bodies, Wang *et al* [105] have proven the existence of certain minimum numbers of orthogonal relative equilibria. We provide a clarification of their result along with a few minor extensions. We then examine the implications of reflective symmetry on our stability analysis. However, we first expand upon the concept of a *constrained potential surface* introduced in Ref. [104] as a means of identifying orthogonal relative equilibria of symmetric rigid bodies.

8.2.1 Constrained Potential Surface. Recall the orthogonal relative equilibrium condition given in Equation (8.2). Equation (8.2a) is easily seen to be the first-order conditions for the variational problem

$$\text{Make stationary } V(\mathbf{\Lambda}) \text{ subject to } \frac{1}{2}|\mathbf{\Lambda}|^2 = c_o. \tag{8.7}$$

where $|\mathbf{\Omega}_e|^2$ acts as the Lagrange multiplier. (Wang *et al* [105] have shown that this Lagrange multiplier will be positive for the physically relevant situation where the center of attraction does not lie within the rigid body.) Fix the constant $c_o > 0$. The set of all $\mathbf{\Lambda}$ satisfying the constraint in variational problem (8.7) is the sphere, $\mathcal{S}$, of radius $\sqrt{2c_o}$. For each $\mathbf{\Lambda}$ on the sphere, compute the corresponding value of the potential (for a given rigid body) and associate this scalar quantity with the $-\mathbf{\Lambda}$ direction to generate a new vector.[1] In this way, the sphere, $\mathcal{S}$, is mapped to a new surface $\mathcal{V}$. This *constrained potential surface* is shown conceptually in Figure 8.2. Critical points of this surface correspond directly to solutions of the variational problem in Equation (8.7). The surface is fixed relative to the body. When a principal axis of the body is orthogonal to the direction of a critical point of $\mathcal{V}$, then there are two solutions to the orthogonal relative equilibrium condition

[1] It is convenient to talk in terms of $-\mathbf{\Lambda}$ rather than $\mathbf{\Lambda}$ in the body frame. This may be thought of as the position vector of the center of attraction which lies somewhere on the constraint sphere. Also, since the potential is negative, for plotting we add a sufficiently large positive constant to make the range of values nonnegative. For instance, we may plot $U(\mathbf{\Lambda}) = V(\mathbf{\Lambda}) - V_{\min}$ where $V_{min} = \min_{\mathbf{\Lambda}\in\mathcal{S}} V(\mathbf{\Lambda})$.

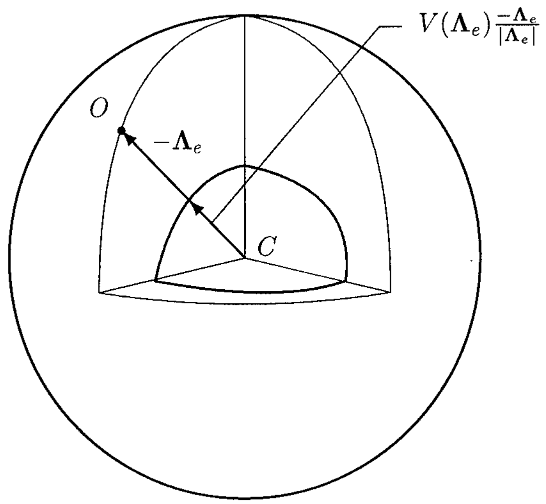


Figure 8.2 Constrained Potential Surface Associated with Constraint Sphere $\mathbb{S}$

corresponding to $-\mathbf{\Lambda}_e$ in the critical point direction and $\mathbf{\Omega}_e$ in either direction along the principal axis. In the example body examined later in this chapter, this surface provides a helpful visualization of orthogonal relative equilibria.

8.2.2 Existence of Orthogonal Relative Equilibria. Wang *et al* [105] proved the existence of orthogonal relative equilibria for rigid bodies with one or more planes of symmetry. Their result is stated as

Theorem 8.1 (WMK Symmetry Theorem). *For a rigid body having a plane of symmetry, there are at least four orthogonal relative equilibria. Furthermore, if the rigid body is symmetric with respect to two planes, there are at least eight orthogonal relative equilibria, and for a rigid body with three planes of symmetry, there are at least 24 orthogonal relative equilibria.*

Note that a plane of symmetry of the rigid body is also a plane of symmetry of the constrained potential surface and the component of ∇V normal to the plane is zero. Then the above result follows from the argument that at least two critical points of the constrained potential surface lie in the plane of symmetry (a maximum and a minimum). A symmetry plane is also a principal plane, thus the axis through the center of mass normal

to this plane is a principal axis. Then there are two choices for $\mathbf{\Lambda}_e$ (the critical point directions) and for each choice of $\mathbf{\Lambda}_e$, two choices of $\mathbf{\Omega}_e$ (either direction along the normal principal axis). The multiple plane results follow with the additional realization that the intersection of two symmetry planes is a principal axis and each direction along this axis must correspond to a critical point of the constrained potential surface. However, implicit in the discussion of the three-symmetry-plane result was the assumption that the planes are all orthogonal. In the case where the three planes intersect in a line (and for any body with more than three planes of symmetry this must occur), there are an infinite number of relative equilibria. This follows from the requirement that if two distinct symmetry planes intersect at other than a right angle the inertias are equal in the principal plane transverse to their axis of intersection, thus providing an infinite number of choices for the direction of $\mathbf{\Omega}_e$. A similar analysis can be made for bodies with rotational symmetries.

8.2.3 Symmetry Implications on Stability. We now return to our stability analysis to consider relative equilibria identified by the WMK Symmetry Theorem. Below we examine two particular cases — a rigid body with one symmetry plane (Case 1) and a rigid body with two symmetry planes which intersect at right angles (Case 2). All other possibilities are simple extensions of these two cases.

Case 1: One Symmetry Plane. For these relative equilibria, the angular velocity is normal to the symmetry plane. This implies the $\hat{\mathbf{b}}_1$-$\hat{\mathbf{b}}_3$ plane is the symmetry plane. Furthermore, any integral over the body of a function which is odd in the $\hat{\mathbf{b}}_2$ coordinate must be zero. In particular, I_{12}, I_{23}, v_{12}, and v_{23} are all zero. The first two are zero by the orthogonal relative equilibrium conditions. The zero requirement on the latter two is new and has an important effect on the spectral stability analysis. Setting these two values to zero in the linear system matrix of Equation (8.4) results in a decoupling of the characteristic equation. That is,

$$P_1(s) = s(s^4 + B_2 s^2 + B_0)(s^4 + C_2 s^2 + C_0) \tag{8.8}$$

Table 8.2 Spectral Stability Conditions — Symmetric Body

$B_0 > 0$	$C_0 > 0$
$B_2 > 0$	$C_2 > 0$
$B_2^2 - 4B_0 > 0$	$C_2^2 - 4C_0 > 0$

$$B_0 = \omega^2 \left\{ v_{22} \left[\Delta + I_2 \left(2I_2 - 1 \right) \right] + R^2 \left(I_2 - I_{11} \right) \left(v_{22} - \omega^2 \right) \right\} / \Delta$$
$$B_2 = \left[\omega^2 I_2 \left(2I_2 - 1 \right) + R^2 I_{33} \left(v_{22} - \omega^2 \right) + \Delta \left(v_{22} + \omega^2 \right) \right] / \Delta$$
$$C_0 = \left\{ I_2 \left[\left(v_{11} - \omega^2 \right) \left(v_{33} - \omega^2 \right) - {v_{13}}^2 \right] + R^2 \left[\left(v_{11} - \omega^2 \right) \left(v_{33} + 3\omega^2 \right) - {v_{13}}^2 \right] \right\} / I_2$$
$$C_2 = \left[R^2 \left(v_{11} - \omega^2 \right) + I_2 \left(v_{11} + v_{33} + 2\omega^2 \right) \right] / I_2$$
$$\Delta = I_{11} I_{33} - I_{13}^2$$

The spectral stability requirements are greatly simplified as shown in Table 8.2. The decoupled systems correspond to a longitudinal subsystem and a lateral-directional subsystem. Similarly, for the nonlinear stability analysis, the projected Hessian decouples and the computation of the stability conditions is simplified. The pivots for the projected Hessian are given in Table 8.3.

Case 2: Two Perpendicular Symmetry Planes. For this case, the radius vector is directed along the axis of intersection of the two planes and the angular velocity is normal to one plane (and lies in the other). Both the $\hat{\mathbf{b}}_1$-$\hat{\mathbf{b}}_3$ and $\hat{\mathbf{b}}_2$-$\hat{\mathbf{b}}_3$ planes are symmetry planes. Thus, any integral over the body of a function which is odd in either the $\hat{\mathbf{b}}_1$ or $\hat{\mathbf{b}}_2$ coordinate is zero. In addition to I_{12}, I_{23}, v_{12}, and v_{23} being zero as above, we also find that I_{13} and v_{13} are zero. Thus, these are principal relative equilibria. Although no further decoupling occurs in the characteristic polynomial, the coefficients and stability conditions in Tables 8.2 and 8.3 are further simplified.

8.3 Example Problem: Symmetric Molecule

To explore the utility of the preceding discussion, we examine the relative equilibria of a very simple rigid body — the symmetric molecule. The molecule consists of six point masses located in pairs along orthogonal axes such that the center of mass lies at the

Table 8.3 Pivots of the Projected Hessian Matrix $\mathbf{P}(\mathbf{z}_e)\nabla^2 F(\mathbf{z}_e)\mathbf{P}(\mathbf{z}_e)$ for a Body with a Plane of Symmetry

Longitudinal System: $(\Sigma_1, \Sigma_3, \Lambda_1, \Lambda_3, \Pi_2)$
$\frac{I_2+R^2+4I_2Rv_3+3I_2R^2v_3^2+I_2R^3v_3v_{33}}{I_2(1+R^2+Rv_3)^2}$
1
$\frac{I_2Rv_{11}+R^3v_{11}-I_2v_3-R^2v_3+4I_2R^2v_{11}v_3-I_2R^4v_{13}^2v_3-4I_2Rv_3^2+3I_2R^3v_{11}v_3^2-3I_2R^2v_3^3+I_2R^4v_{11}v_3v_{33}-I_2R^3v_3^2v_{33}}{R(I_2+R^2+4I_2Rv_3+3I_2R^2v_3^2+I_2R^3v_3v_{33})}$
$\frac{I_2R^2v_{13}^2+R^4v_{13}^2+I_2Rv_{11}v_3-3R^3v_{11}v_3-I_2v_3^2+3R^2v_3^2-I_2R^2v_{11}v_{33}-R^4v_{11}v_{33}+I_2Rv_3v_{33}+R^3v_3v_{33}}{R(-I_2Rv_{11}-R^3v_{11}+I_2v_3+R^2v_3-4I_2R^2v_{11}v_3+I_2R^4v_{13}^2v_3+4I_2Rv_3^2-3I_2R^3v_{11}v_3^2+3I_2R^2v_3^3-I_2R^4v_{11}v_3v_{33}+I_2R^3v_3^2v_{33}}$
0
Lateral-Directional System: $(\Sigma_2, \Lambda_2, \Pi_1, \Pi_3)$
$\frac{I_2}{I_2+R^2}$
$\frac{I_2v_{22}+R^2v_{22}-Rv_3}{I_2+R^2}$
$\frac{I_2I_{33}-\Delta}{I_2\Delta}$
$\frac{-I_{13}^2I_2^2v_{22}+I_{11}I_2^2I_{33}v_{22}-I_{13}^2I_2R^2v_{22}+I_{11}I_2I_{33}R^2v_{22}+I_{13}^2I_2Rv_3-I_{11}I_2I_{33}Rv_3-I_{11}I_2v_{22}\Delta-I_2I_{33}v_{22}\Delta-I_{11}R^2v_{22}\Delta+I_{11}Rv_3\Delta+v_{22}\Delta^2}{\Delta(I_2v_{22}+R^2v_{22}-Rv_3)(I_2I_{33}-\Delta)}$

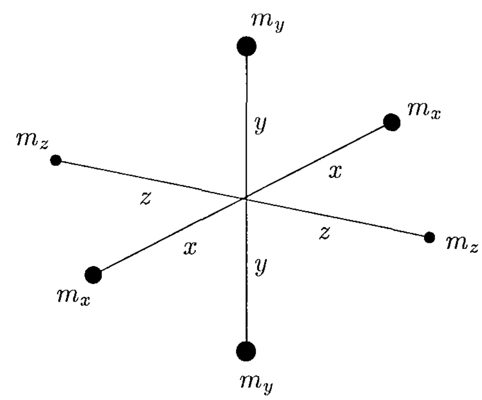


Figure 8.3 The Symmetric Molecule

intersection of the axes. We consider the symmetric molecule where each pair consists of two equal masses (necessarily located at equal distances from the center of mass). The general configuration is shown in Figure 8.3. The symmetric molecule has been examined previously by Meirovitch [69] for the case of three equal inertias and by Beletskii [15] for the case of distinct inertias. Wang *et al* [105] have investigated the asymmetric molecule.

8.3.1 Principal Relative Equilibria. The symmetric molecule has three orthogonal symmetry planes with each pair of point masses located along an axis of intersection of two of these planes. These axes are principal axes and considering Case 2 above we conclude that all twenty-four principal relative equilibria exist for this body.

The discrete nature of the point mass model simplifies our analysis by reducing body integrals to summations. The potential function for the symmetric molecule is

$$V(\mathbf{\Lambda}) = -\sum_{i=1}^{6} \frac{m_i}{|\mathbf{\Lambda} + \mathbf{a}_i|} \tag{8.9}$$

and the resulting gradient and Hessian are

$$\nabla V(\mathbf{\Lambda}) = \sum_{i=1}^{6} \frac{m_i(\mathbf{\Lambda} + \mathbf{a}_i)}{|\mathbf{\Lambda} + \mathbf{a}_i|^3} \tag{8.10}$$

and

$$\nabla^2 V(\mathbf{\Lambda}) = \sum_{i=1}^{6} \frac{m_i}{|\mathbf{\Lambda}+\mathbf{a}_i|^5}\left[|\mathbf{\Lambda}+\mathbf{a}_i|^2\,\mathbf{1} - 3(\mathbf{\Lambda}+\mathbf{a}_i)(\mathbf{\Lambda}+\mathbf{a}_i)^{\mathsf{T}}\right]. \tag{8.11}$$

For principal relative equilibria in the body frame, only the $\hat{\mathbf{b}}_3$-component of ∇V is nonzero:

$$v_3 = -2R\left[\frac{m_x}{(R^2+x^2)^{3/2}} + \frac{m_y}{(R^2+y^2)^{3/2}} + \frac{m_z(R^2+z^2)}{R(R^2-z^2)^2}\right]. \tag{8.12}$$

The corresponding magnitude of the angular velocity is given by Equation (8.3c) as

$$\omega = \left\{2\left[\frac{m_x}{(R^2+x^2)^{3/2}} + \frac{m_y}{(R^2+y^2)^{3/2}} + \frac{m_z(R^2+z^2)}{R(R^2-z^2)^2}\right]\right\}^{1/2}. \tag{8.13}$$

The Hessian for this case is diagonal with

$$v_{11} = 2\left[\frac{m_x(R^2-2x^2)}{(R^2+x^2)^{5/2}} + \frac{m_y}{(R^2+y^2)^{3/2}} + \frac{m_z R(R^2+3z^2)}{(R^2-z^2)^3}\right] \tag{8.14a}$$

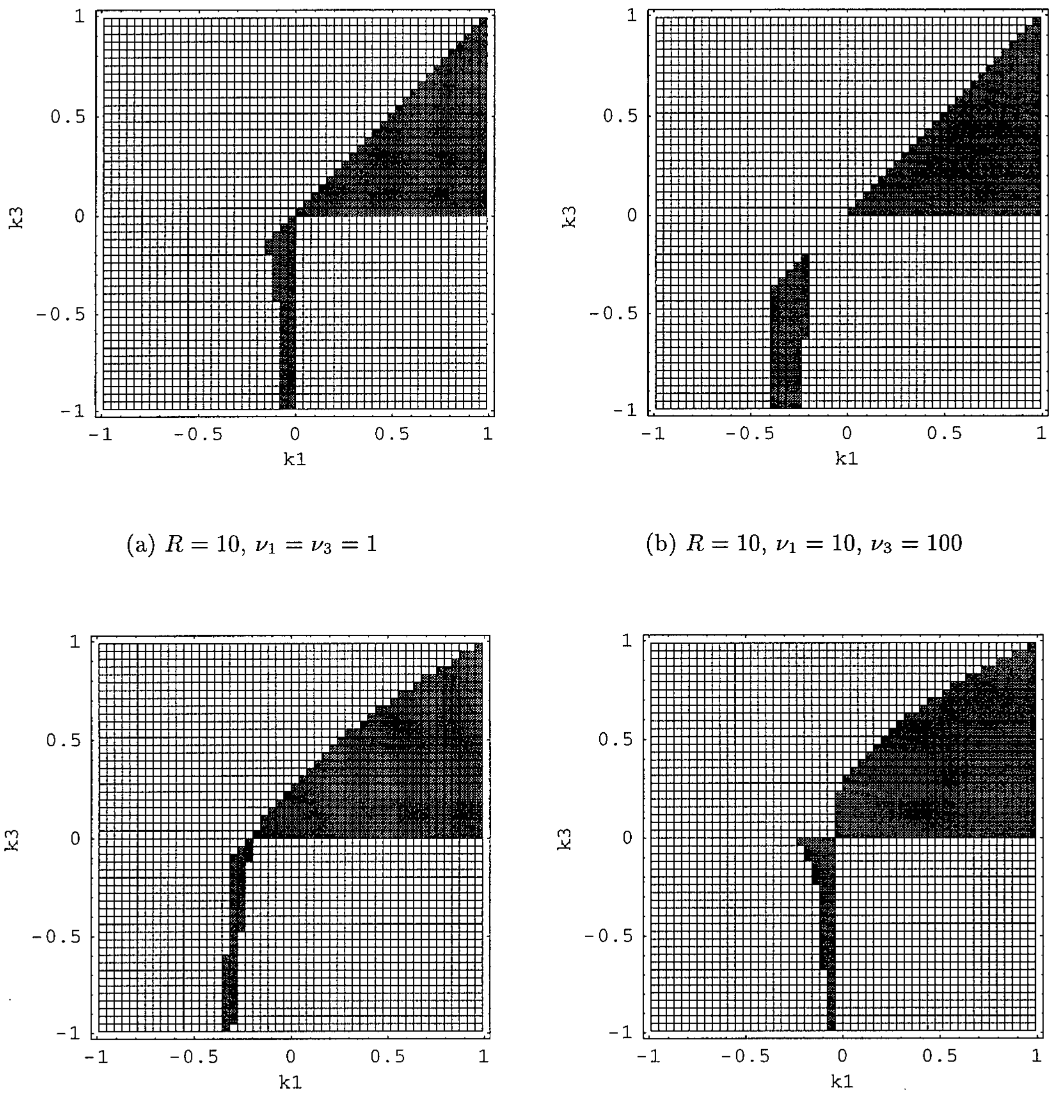
$$v_{22} = 2\left[\frac{m_x}{(R^2+x^2)^{3/2}} + \frac{m_y(R^2-2y^2)}{(R^2+y^2)^{5/2}} + \frac{m_z R(R^2+3z^2)}{(R^2-z^2)^3}\right] \tag{8.14b}$$

$$v_{33} = 2\left[\frac{m_x(-2R^2+x^2)}{(R^2+x^2)^{5/2}} + \frac{m_y(-2R^2+y^2)}{(R^2+y^2)^{5/2}} + \frac{-2m_z R(R^2+3z^2)}{(R^2-z^2)^3}\right] \tag{8.14c}$$

These expressions may be applied to the conditions in Table 8.2 to determine the linear stability of the relative equilibrium.

The configuration of the (nondimensional) molecule is described by the mass ratios $\nu_1 = m_x/m_y$ and $\nu_3 = m_z/m_y$ and the Smelt inertia parameters k_1 and k_3. Figure 8.4 shows the spectral stability regions at several different values of ν_1 and ν_3 for a small radius, R. The Lagrange region is essentially preserved while the DeBra-Delp region shifts with variations in the mass ratios. Note that some of these configurations are spectrally stable when the rotation associated with orbital motion is about the intermediate axis.

8.3.2 Additional Orthogonal Relative Equilibria. Figure 8.5 shows contour plots of the constrained potential surface for several different values of k_1 and k_3 with R, ν_1, and



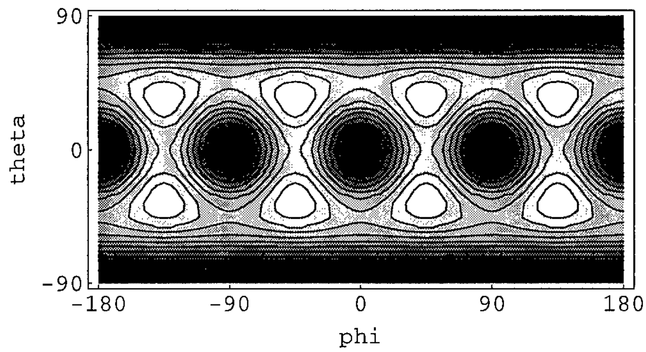
(a) $R = 10$, $\nu_1 = \nu_3 = 1$

(b) $R = 10$, $\nu_1 = 10$, $\nu_3 = 100$

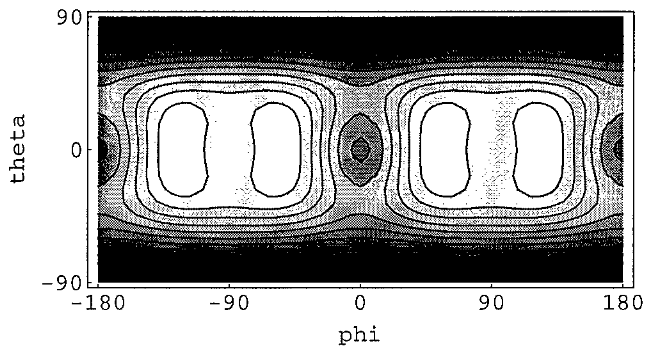
(c) $R = 10$, $\nu_1 = 10$, $\nu_3 = 0.1$

(d) $R = 10$, $\nu_1 = 0.01$, $\nu_3 = 0.1$

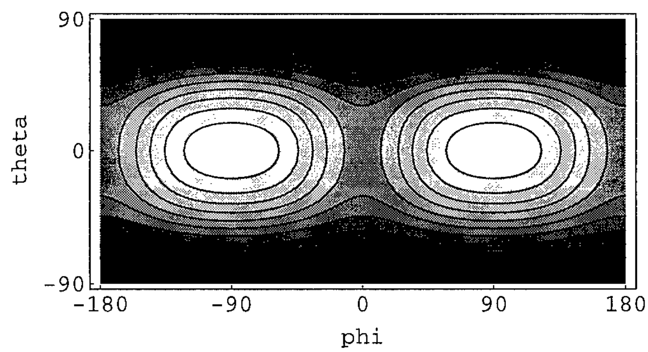
Figure 8.4 Spectral Stability of Principal Relative Equilibria for the Symmetric Molecule at Small Radius (gray=spectrally stable, white=unstable)



(a) $k_1 = 0$ and $k_3 = 0$



(b) $k_1 = 0.01$ and $k_3 = 0.005$



(c) $k_1 = 0.04$ and $k_3 = 0.02$

Figure 8.5 Contour Plots of Constrained Potential Surface for Varying Inertia Ratios ($\nu_1 = \nu_3 = 1$ and $R = 10$; white=maximum, black=minimum)

ν_3 fixed. The distance to the surface is plotted as a function of the angles defined in Figure 8.6 for a specific principal axis frame. From these contour plots it is apparent that the six principal axis directions (including signs) are not necessarily the only critical point directions on the constrained potential surface. We can eliminate the maxima (lighter shaded regions) as possible directions for $\mathbf{\Lambda}_e$ (except when two inertias are equal) since there is no corresponding choice for $\mathbf{\Omega}_e$. (This does not rule out the possibility of oblique relative equilibria associated with this direction.) The saddles, on the other hand, do correspond to relative equilibria. Since they lie in a symmetry plane, the relative equilibria must satisfy the conditions described in Case 1 of the previous section. Returning to the body frame used previously, we consider the situation shown in Figure 8.7. Equation (8.3a) gives

$$m_x x \cos\alpha \left[\frac{1}{(R^2 + x^2 - 2Rx\sin\alpha)^{3/2}} - \frac{1}{(R^2 + x^2 + 2Rx\sin\alpha)^{3/2}} \right] = m_z z \sin\alpha \left[\frac{1}{(R^2 + z^2 - 2Rz\cos\alpha)^{3/2}} - \frac{1}{(R^2 + z^2 + 2Rz\cos\alpha)^{3/2}} \right]. \quad (8.15)$$

For all configurations, $\alpha = n\pi/2$, $n \in \mathbb{Z}$, is a solution to this transcendental equation. This solution corresponds to the principal relative equilibria described above. For configurations where other solutions exist to this equation, there are orthogonal relative equilibria that are not principal relative equilibria. This is clearly in contrast to the classical result based on the second-order approximation. Figure 8.7 suggests that these correspond to a state of balance between the forces in the $\hat{\mathbf{b}}_1$ direction. By expanding both sides of Equation (8.15) in a series, we find that to first order this equation is satisfied when $I_1 = I_3$. Numerical investigations confirm that these relative equilibria only exist when the two inertias are very nearly equal.

8.4 Conclusions

In this chapter, we have explored the relative equilibrium conditions for the free rigid-body system when the exact form of the potential is used. While general results were

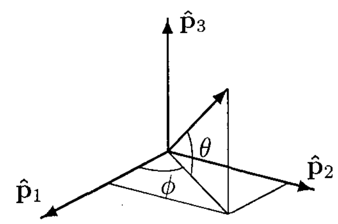


Figure 8.6 Definition of Angles Used in Contour Plots

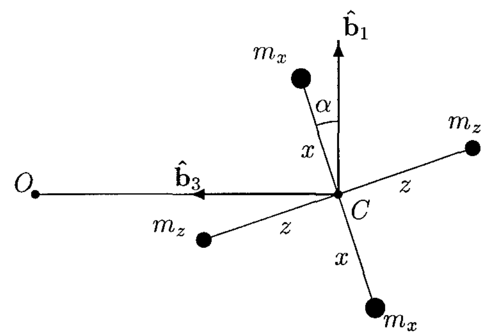


Figure 8.7 Configuration in the Orbit Plane for Non-Principal Orthogonal Relative Equilibria

limited, in the presence of one or more planes of reflective symmetry in the body we found that the classical results could again be extended. We also found that the exact potential allows for the possibility of other orthogonal relative equilibria to exist.

IX. Summary and Conclusions

In this chapter, we review the findings of this dissertation and discuss proposals for future work to extend these results.

9.1 Summary

In the preceding analysis, we have applied a noncanonical Hamiltonian approach to examine the relative equilibria of a rigid body moving in a central gravitational field. These equilibria correspond to fixed points of a reduced set of equations of motion expressed in a rotating frame. This motion is representative of a rigid satellite moving in circular orbit about a spherical primary with a fixed attitude relative to an observer rotating at the orbital rate.

Our objective was to clarify the relationship between the classical approximation dating back to Lagrange [52], in which the series expansion of the force and torque are truncated after the lead term, and the noncanonical Hamiltonian treatment of Wang *et al* [104, 105]. In the classical approximation, the orbital equations are independent of the attitude motion and take the same form as the Keplerian two-particle problem. Hence, the center of mass of the body moves in a circular orbit about the center of attraction with Keplerian frequency. For a rigid body with distinct inertias, the classical approximation has twenty-four relative equilibrium solutions — the principal relative equilibria — for which the attitude of the body is such that the principal inertia axes are aligned with the radial, tangential, and orbit normal directions. On the other hand, Wang *et al* [105] used a noncanonical formulation with approximations derived from truncated series expansions of the potential. In this system, the orbital and attitude equations of motion remain coupled for all but the trivial zeroth-order approximation. They have proven that the general solution has a circular orbit for which the orbit center and the center of attraction lie on the axis of rotation but are not necessarily coincident as in the classical approximation. We introduced the terms *orthogonal* and *oblique* to refer to solutions where the centers are and are not coincident,

respectively. Wang *et al* also showed that for certain asymmetric bodies no orthogonal relative equilibria exist and the orbits must be oblique.

Our approach involved development of a hierarchy of noncanonical Hamiltonian approximations for rigid body motion in a central gravitational field. The hierarchy consists of the WMK system (of Wang *et al* [104, 105]) and two new noncanonical formulations which we derived for rigid bodies subject to certain constraints — motion about a point fixed in inertial space and motion about a point which follows a Keplerian orbit. We demonstrated that the classical solution is dynamically equivalent to this latter constrained (Keplerian) system with a second-order potential approximation. We then applied modern Hamiltonian methods to identify the relative equilibria and determine conditions for spectral and nonlinear stability. The results for the classical approximation are well known and served to validate our techniques. We applied these same methods to the WMK system (which represents unconstrained or free rigid-body motion) with a second-order potential approximation. In this manner, we were able to examine more closely the similarities and differences of these two systems.

In addition to the tri-inertial rigid body, we also considered relative equilibria of an axisymmetric rigid body for both the Keplerian and WMK systems. Relative equilibrium solutions and conditions for stability of the classical approximation are also well known in this instance. However, the axisymmetric case of the WMK system had not previously been examined. We again applied Hamiltonian methods to determine the classes of relative equilibria for both systems and determined conditions for stability of most of these solutions.

Finally, we followed up on the investigation of Wang *et al* [105] into the WMK system with the exact form of the potential. We focused on orthogonal relative equilibria and the requirements placed on the inertias and potential of the body for these to exist. We discussed bodies with reflective symmetries for which these requirements are satisfied and the orthogonal relative equilibria are guaranteed to exist. We considered an example body with reflective symmetry and showed that, whereas the principal relative equilibria always

exist in this circumstance, the equations of motion allow additional orthogonal solutions for which the principal inertia axes in the orbit plane are not aligned with the radial and tangential directions.

9.2 Conclusions

The noncanonical formulations of the Keplerian and WMK systems have considerable differences. While the WMK system is derived through reduction of a twelfth-order canonical system, the Keplerian system is associated with a sixth-order canonical system which must be inflated to incorporate constraints. Both resulting noncanonical systems are ninth-order. The Keplerian system has three Casimir functions which are all trivial relationships between orthogonal unit vectors, while the WMK system has only one Casimir function, but it is physically meaningful (conservation of angular momentum about center of attraction).

In spite of these differences, there are striking similarities in the form of relative equilibrium conditions and the resulting solutions. Table 9.1 shows the Likins-Roberson and Pringle-Likins Conditions for each system. We can see that the unit vectors in the orbit normal and radial directions, $\boldsymbol{\beta}_e$ and $\boldsymbol{\gamma}_e$, in the Keplerian system play a similar role to the angular velocity and radius, $\boldsymbol{\Omega}_e$ (or $\widetilde{\boldsymbol{\Omega}}_e$) and $\boldsymbol{\Lambda}_e$, in the second-order WMK system. When we consider the exact form of the WMK system, it becomes apparent that $3\mathbf{I}\boldsymbol{\gamma}_e$ and $3\mathbf{I}\boldsymbol{\Lambda}_e/\left|\boldsymbol{\Lambda}_e\right|^5$ in the other two systems are actually playing the same role as $\nabla V(\boldsymbol{\Lambda}_e)$ in the exact system.

Focusing on the arbitrary body, the Likins-Roberson Condition for each system may be used to prove that four certain vectors are all coplanar. For the Keplerian system, these vectors are $\boldsymbol{\beta}_e$, $\boldsymbol{\gamma}_e$,$\mathbf{I}\boldsymbol{\beta}_e$, and $\mathbf{I}\boldsymbol{\gamma}_e$. Because $\boldsymbol{\beta}_e$ and $\boldsymbol{\gamma}_e$ must be orthogonal, this can then be used to prove that any relative equilibrium must be a principal relative equilibrium. For the WMK system, the absence of the orthogonality requirement prevents this same conclusion from being drawn. However, for the second-order system, we are still able to prove this to be true for all but extremely small values of $|\boldsymbol{\Lambda}_e|$. The existence of oblique

Table 9.1 Relative Equilibrium Conditions for Keplerian and WMK Systems

	Arbitrary (Likins-Roberson Condition)	Axisymmetric (Pringle-Likins Condition)
Keplerian (2nd-order)	$3\gamma_e^{\times}\mathbf{I}\gamma_e - \boldsymbol{\beta}_e^{\times}\mathbf{I}\boldsymbol{\beta}_e = \mathbf{0}$	$3\gamma_e^{\times}\mathbf{I}\gamma_e - \boldsymbol{\beta}_e^{\times}\mathbf{I}\boldsymbol{\beta}_e - \mu_4\boldsymbol{\beta}_e^{\times}\mathbf{I}\mathbf{1}_3 = \mathbf{0}$
WMK (2nd-order)	$\frac{3}{\lvert\boldsymbol{\Lambda}_e\rvert^5}\boldsymbol{\Lambda}_e^{\times}\mathbf{I}\boldsymbol{\Lambda}_e - \boldsymbol{\Omega}_e^{\times}\mathbf{I}\boldsymbol{\Omega}_e = \mathbf{0}$	$\frac{3}{\lvert\boldsymbol{\Lambda}_e\rvert^5}\boldsymbol{\Lambda}_e^{\times}\mathbf{I}\boldsymbol{\Lambda}_e - \widetilde{\boldsymbol{\Omega}}_e^{\times}\mathbf{I}\widetilde{\boldsymbol{\Omega}}_e - \mu_2\widetilde{\boldsymbol{\Omega}}_e^{\times}\mathbf{I}\mathbf{1}_3 = \mathbf{0}$
WMK (exact)	$\boldsymbol{\Lambda}_e^{\times}\nabla V(\boldsymbol{\Lambda}_e) - \boldsymbol{\Omega}_e^{\times}\mathbf{I}\boldsymbol{\Omega}_e = \mathbf{0}$	$\boldsymbol{\Lambda}_e^{\times}\nabla V(\boldsymbol{\Lambda}_e) - \widetilde{\boldsymbol{\Omega}}_e^{\times}\mathbf{I}\widetilde{\boldsymbol{\Omega}}_e - \mu_2\widetilde{\boldsymbol{\Omega}}_e^{\times}\mathbf{I}\mathbf{1}_3 = \mathbf{0}$

relative equilibria for the exact WMK system demonstrates that the uniqueness of the principal relative equilibria in the other systems is a result of symmetries implicit in the second-order potential approximation, and not a result of the Keplerian constraint. For bodies in which these symmetries are present, the classical results readily extend to the exact WMK formulation. Significant deviations from the classical results occur only at small distances or for bodies with unusual, impractical configurations (such as a gravity gradient satellite with two nearly equal inertias). Hence, for general analysis of relative equilibria of a rigid body, the Keplerian approximation is quite sufficient.

9.3 Recommendations

Our investigations focused strictly on the relative equilibria for a rigid body in a central gravitational field. For these particular types of solutions to the equations of motion, the Keplerian system provided accurate results for all practical purposes. We may draw no such conclusion regarding the general motion of a rigid body. Since the Poisson structure and the conserved quantities of the Keplerian system are considerably different from those of the WMK system, there is no reason to expect the dynamics to be similar. In particular, studies of chaotic motion, which one would expect to be extremely sensitive to system structure, might lead to different results when orbit-attitude coupling is considered. An investigation into dynamics of the WMK system other than relative equilibrium would

provide insight into this question. Also, generalization of the Keplerian system to orbits other than circular might prove useful.

The rigid body may be a reasonable approximation for passive gravity-gradient satellites. However, most satellites use active control, often in the form of momentum control devices, to maintain or change attitude. Hence, one might want to extend the WMK system to consider a gyrostat in a central gravitational field. In hindsight, this might have been a better approach than the one taken here since the arbitrary and axisymmetric rigid body problems should just be limiting cases of the gyrostat problem as the rotor or platform mass goes to zero.

In Chapter 6, we applied several nonlinear stability analysis techniques which are based upon Hamiltonian methods. The projection method was quite successful and far surpassed the other techniques in terms of simplicity. Therefore, all subsequent nonlinear stability analyses were done using this approach. The projection method is algorithmic and could be implemented to perform nonlinear stability analyses without user intervention once the Hamiltonian and Casimir functions have been specified. The development of such general algorithms should be investigated. In addition, for the symmetric body, we eliminated cyclic perturbations by revising our definition of stability. However, by doing so, we are no longer seeking the same constrained minimum and it is not apparent that the theorem which guarantees the existence of a Liapunov function still applies. The mathematical details of this argument should be refined.

Appendix A. A Primer on Hamiltonian Mechanics

In this appendix, we present a brief review of Hamiltonian systems. Starting with canonical systems, we examine the equations of motion, Poisson brackets, and first integrals. We then show how these concepts generalize to noncanonical systems and discuss the special first integrals known as Casimir functions. The last section identifies sources for further reading.

A.1 Canonical Systems

A Hamiltonian system is a mechanical system for which the equations of motion may be expressed in the canonical form

$$\dot{q}_i = \frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H(\mathbf{q},\mathbf{p},t)}{\partial q_i} \qquad \text{for } i = 1, 2, \ldots, n \tag{A.1}$$

where $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ are the *generalized coordinates* and *conjugate momenta* and $H(\mathbf{q},\mathbf{p},t)$ is the *Hamiltonian.* These equations are often referred to as *Hamilton's Equations.* The coordinates and momenta are collectively known as the *phase variables* and the $2n$-dimensional space of phase variables is the *phase space.* Introducing the phase variable vector $\mathbf{z} = (\mathbf{q},\mathbf{p})$, the equations of motion take the compact form

$$\dot{\mathbf{z}} = \mathbf{J}\,\nabla H(\mathbf{z},t) \tag{A.2}$$

where $\mathbf{J}$ is the $2n \times 2n$ symplectic matrix

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix} \tag{A.3}$$

and ∇ denotes the gradient (with respect to the phase variable vector).

An alternative representation of the system is given in terms of the Poisson bracket operator. Let $F(\mathbf{q},\mathbf{p},t)$ and $G(\mathbf{q},\mathbf{p},t)$ be smooth functions of their arguments. Then we

define the Poisson bracket of F and G as

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}. \tag{A.4}$$

The Poisson bracket is a skew-symmetric and bilinear operator. In terms of the phase variable vector, $\mathbf{z}$, the Poisson bracket of $F(\mathbf{z}, t)$ and $G(\mathbf{z}, t)$ is

$$\{F, G\} = \nabla F(\mathbf{z}, t) \cdot \mathbf{J}\, \nabla G(\mathbf{z}, t). \tag{A.5}$$

Associated with every Hamiltonian is a vector field defined by

$$\hat{\mathbf{v}}_H(F) = \{F, H\}. \tag{A.6}$$

The flow of this vector field in phase space is governed by Hamilton's Equations (A.1) and represent trajectories of the Hamiltonian system.

The time derivative of a smooth function F is

$$\begin{aligned} \frac{d}{dt}F(\mathbf{z}, t) &= \nabla F(\mathbf{z}, t) \cdot \dot{\mathbf{z}} + \frac{\partial F(\mathbf{z}, t)}{\partial t} \\ &= \nabla F(\mathbf{z}, t) \cdot \mathbf{J}\nabla H(\mathbf{z}, t) + \frac{\partial F(\mathbf{z}, t)}{\partial t} \\ &= \{F, H\} + \frac{\partial F(\mathbf{z}, t)}{\partial t}. \end{aligned} \tag{A.7}$$

In particular, for the phase variables themselves, we have

$$\dot{z}_i = \{z_i, H\} \tag{A.8}$$

as an alternative expression of Hamilton's Equations (A.1). Furthermore, for the Hamiltonian we have

$$\frac{d}{dt}H(\mathbf{z}, t) = \{H, H\} + \frac{\partial H(\mathbf{z}, t)}{\partial t}. \tag{A.9}$$

The skew-symmetry of the Poisson bracket assures the first term on the right-hand side must be zero. Therefore, if the Hamiltonian is not an explicit function of time, it must be constant along trajectories in phase space. Hamiltonian systems of this type are referred to as *conservative.* The equations of motion for a conservative system are autonomous.

The Hamiltonian of a conservative system is just one example of a class of special functions known as *integrals of motion* or *first integrals* which are constant along trajectories of the system. A first integral $C(\mathbf{z}, t)$ must therefore satisfy the equation

$$\{C, H\} = -\frac{\partial C(\mathbf{z}, t)}{\partial t}. \tag{A.10}$$

The level surfaces of C are known as *invariant sets* since a trajectory which begins in an invariant set remains in that set for all time. The invariant set is typically a manifold of dimension $2n - 1$. Knowledge of $2n - 1$ independent first integrals would allow us to completely specify the solution. A system which can be solved in this manner is *integrable.*

If the first integral C is not an explicit function of time, then we find $\{C, H\} = 0$. Two functions which have a Poisson bracket identically equal to zero are said to be in *involution*, or sometimes are said to be *J-orthogonal.* A set of n first integrals which are linearly independent and pairwise in involution form a *Lagrangian set.* Knowledge of a Lagrangian set is sufficient to guarantee a system is integrable.[1]

Symmetries play a special role in Hamiltonian systems. A coordinate is *cyclic* or *ignorable* if it does not appear in the Hamiltonian. An immediate implication of this is that the momentum conjugate to this coordinate is an integral of motion. Cyclic coordinates are associated with a symmetry of the system. A theorem due to Noether[2] identifies a correspondence between symmetries and first integrals. Thus, knowledge of symmetries enables us to identify integrals.

[1]See Theorem 6 of Section II.A in Meyer and Hall [71].
[2]See, *e.g.*, Olver [79:410].

A.2 Noncanonical Systems

For canonical systems, the phase space is $\mathbb{R}^{2n}$. However, it is often convenient to work with variables defined on other manifolds. A prime example is the rotation of a rigid body. Classically, the rotation is described by a direction cosine matrix parameterized by Euler angles which form a set of three generalized coordinates. However, the use of the Euler angles can result in considerable algebraic manipulation involving trigonometric functions and can lead to singularites at certain attitudes. It can be considerably simpler to speak generically in terms of a direction cosine matrix without parameterizing the matrix. The direction cosine matrix is an element of the group $SO(3)$ of orthogonal matrices with determinant $+1$. This group forms a three-dimensional manifold embedded in the nine-dimensional space $\mathbb{R}^{3\times 3}$. In the main text, we find that analysis involving the direction cosine matrix directly can lead to more general results which are not masked by the intricacies of a particular parameterization and which allow direct substitution of any parameterization.

The manifold $SO(3)$ is an example of a *Poisson manifold.* A Poisson manifold is a smooth manifold with a Poisson bracket defined on it. The Poisson bracket we refer to here is a generalization of the Poisson bracket defined above for the canonical system. It is a transformation which maps two smooth functions defined on the manifold to a third smooth function such that the following properties are satisfied:

(i)	$\{F,G\} = -\{G,F\}$	(Skew-Symmetry)
(ii)	$\{c_1F + c_2G, H\} = c_1\{F,H\} + c_2\{G,H\}$ and	(Bilinearity)
	$\{F, c_1G + c_2H\} = c_1\{F,G\} + c_2\{F,H\}$	
(iii)	$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0$	(Jacobi Identity)
(iv)	$\{F, G\cdot H\} = \{F,G\}\cdot H + G\cdot\{F,H\}$	(Leibniz' Rule)

for three smooth functions F, G, and H on the manifold M. It can be shown [79:393–394] that the Poisson bracket can be expressed as

$$\{F, G\} = \nabla F(\mathbf{z}) \cdot \mathbf{J}(\mathbf{z}) \nabla G(\mathbf{z}) \tag{A.11}$$

where $\mathbf{J}(\mathbf{z})$ is a skew-symmetric matrix called the *Poisson structure matrix* of M. Most of what was presented above for canonical systems extends directly to these systems with the substitution of the structure matrix $\mathbf{J}(\mathbf{z})$ for the standard symplectic matrix $\mathbf{J}$. Here, we emphasize the differences.

The canonical system has coordinates and momenta which are conjugate to each other and the phase space must be even-dimensional. No such restriction is placed on noncanonical systems. There is no pairing of phase variables in general, and subsequently the system may have an odd dimension. In addition, the structure matrix is a function of the phase variables and can be nonsingular. This property is very important in that it permits a special class of first integrals known as *Casimir* or *distinguished functions.*[3]

Casimir functions are functions for which the Poisson bracket with any other function is identically zero. That is, for a Casimir function $C(\mathbf{z})$ and any smooth function $F(\mathbf{z}, t)$, we have

$$\{C, F\} = \nabla C(\mathbf{z}) \cdot \mathbf{J}(\mathbf{z}) \nabla F(\mathbf{z}, t) = 0 \tag{A.12}$$

or upon transposing the dot product

$$\nabla F(\mathbf{z}, t) \cdot \mathbf{J}(\mathbf{z}) \nabla C(\mathbf{z}, t) = 0. \tag{A.13}$$

[3]In fact, Casimir functions exist for all systems. However, for a full-rank structure matrix, the only Casimir functions are the constant functions.

Since this latter statement must be valid for any smooth function F defined on M, we must have

$$\mathbf{J}(\mathbf{z})\nabla C(\mathbf{z}) = \mathbf{0}. \tag{A.14}$$

Hence, the gradient of a Casimir function must lie in the null space of the structure matrix. Conversely, any vector in the null space must be the gradient of a Casimir function. If one Casimir function exists, then an infinite number of them exist since, given a Casimir function $C(\mathbf{z})$, any function $C_\phi(\mathbf{z}) = \phi(C(\mathbf{z}))$ (where ϕ is a smooth scalar function) is also a Casimir function. This follows because $\nabla C_\phi(\mathbf{z}) = \phi'(C(\mathbf{z}))\nabla C(\mathbf{z})$. Only a finite number of these functions are of interest. The difference between the order of the Hamiltonian system and the rank of the structure matrix is the number of Casimir functions which have pairwise linearly independent gradients. Such a set forms a basis for the null space of the structure matrix. In practice, this set can be found without much difficulty once the structure matrix has been determined.

If the dimension of the Hamiltonian system is n and there are m linearly independent Casimir functions, then the rank of the structure matrix is $n - m$. The rank is always even and, by a theorem of Darboux [79:405], the system may be represented locally by a canonical system of order $n - m$. Hence, the number of degrees of freedom of the system is given by $(n - m)/2$. While this suggests that a noncanonical system is always of higher order than a canonical system representing the same dynamics, one should not be misled to disregard the utility of noncanonical systems. These systems are often derived from a higher-order canonical system through a process of reduction which eliminates degrees of freedom associated with ignorable coordinates. Although Darboux' theorem assures the existence of a canonical system of equal or lower order, we are not assured that the canonical variables are easy to determine or that this representation is useful.

As used here, the title "noncanonical" is a bit of a misnomer since canonical systems are a subset of noncanonical systems. In much of the literature, no such distinction is

made and all are referred to as Hamiltonian systems. We may do this without changing our definition of Hamiltonian system presented above since Darboux' theorem shows that any noncanonical system may be transformed (locally) into a canonical system. We should also point out that in some of the literature noncanonical systems are referred to as Poisson systems.

A.3 Further Reading

The above discussion is merely a cursory review. It is meant to highlight particular concepts relevant to our study. Canonical Hamiltonian systems are covered in most texts on classical mechanics. See, *e.g.*, Goldstein [30], Marion [66], or Meirovitch [70]. For a very complete, modern treatment, see Meyer and Hall [71]. Noncanonical systems are treated by Olver [79], Scheck [96], Abraham and Marsden [1], and Marsden and Ratiu [67]. The latter two require a firm foundation in differential geometry and group theory.

Appendix B. The Gravitational Potential

This appendix examines the gravitational potential for a closed system of n particles or rigid bodies. It then goes on to explore the idealization of a rigid body in a central gravitational field. The potential for this system is investigated in great detail. A series expansion is presented which allows approximation of the potential to varying degrees of accuracy and complexity. Expressions for the force and torque acting on the rigid body are developed which allow direct use of any order of approximation of the potential. Several of the simpler approximations are discussed.

The results in this appendix are generally well-known. They are presented here to provide a single point of reference in a consistent notation and to allow us to expound upon some of the concepts in greater detail. The potential for a system of particles is developed in every textbook on celestial mechanics. Duboshin [24, 25] presented the generalization to a system of rigid bodies. The potential for a rigid body in a central gravitational field was introduced into the modern astronautical literature by Roberson and Tatistcheff [88]. The development we present (and particularly the approach to deriving consistent expressions for force and torque from the same approximation of the potential) is due to Wang *et al* [104]. We generalize their results by allowing the origin of the body frame to lie at a point other than the center of mass. This leads to the appearance of a first-order term in the force and torque and permits application to a broader class of problems. This generalization was known to Beletskii [15].

B.1 A System of n Particles

Newton's Law of Gravitation states that two particles are attracted by a force which is proportional to each of the masses and inversely proportional to the square of the distance between them. This force acts along the line joining the two particles. For two particles of mass m_i and m_j with position vectors $\boldsymbol{\xi}_i$ and $\boldsymbol{\xi}_j$ in some inertial frame, we may write

$$\mathbf{f}_{ij} = -\frac{Gm_i m_j}{\left|\boldsymbol{\xi}_{ij}\right|^2} \frac{\boldsymbol{\xi}_{ij}}{\left|\boldsymbol{\xi}_{ij}\right|} \tag{B.1}$$

where $\mathbf{f}_{ij}$ is the force on the ith particle due to the jth particle, $\boldsymbol{\xi}_{ij}$ is the position of the ith particle relative to the jth particle ($\boldsymbol{\xi}_{ij} = \boldsymbol{\xi}_i - \boldsymbol{\xi}_j$), and G is the universal gravitational constant. Furthermore, we have $\mathbf{f}_{ji} = -\mathbf{f}_{ij}$. For a closed system of n particles, the force on the ith particle is then

$$\mathbf{f}_i(\boldsymbol{\xi}) = \sum_{\substack{j=0 \\ j\neq i}}^{n-1} \mathbf{f}_{ij}(\boldsymbol{\xi}) \qquad \boldsymbol{\xi} = (\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_{n-1}) \tag{B.2}$$

Since the gravitational forces are only dependent on the relative positions, $\boldsymbol{\xi}_{ij}$, we know that they are conservative.[1] That is, the total work $\sum_{i=0}^{n-1} \mathbf{f}_i \cdot d\boldsymbol{\xi}_i$ is an exact differential of a scalar potential function. In this case, the function is

$$V(\boldsymbol{\xi}) = -\sum_{i<j} \frac{Gm_i m_j}{|\boldsymbol{\xi}_{ij}|}. \tag{B.3}$$

The forces may then be derived from the potential as $\mathbf{f}_i = \nabla_{\xi_i} V(\boldsymbol{\xi})$.

B.2 A System of n Rigid Bodies

If we wish to extend the above result to treat rigid bodies of finite extent, we must consider the gravitational interaction pairwise between particles or elements in different bodies. To do this, we introduce the differential mass element

$$dm_i = \rho_i(\mathbf{a}_i)d\mathbf{a}_i \tag{B.4}$$

where ρ_i is the density distribution of body $\mathcal{B}_i$, $\mathbf{a}_i$ is the position of the element expressed in a frame $\mathcal{F}_{b_i}$ fixed in the body, and $d\mathbf{a}_i$ is a differential volume. For bodies $\mathcal{B}_i$ and $\mathcal{B}_j$, the mutual potential is

$$V_{ij} = -G\int_{\mathcal{B}_i} dm_i \int_{\mathcal{B}_j} dm_j \frac{1}{|\boldsymbol{\xi}_{ij} + \mathbf{B}_i\mathbf{a}_i - \mathbf{B}_j\mathbf{a}_j|} \tag{B.5}$$

[1] See, *e.g.*, Arnold [6:7].

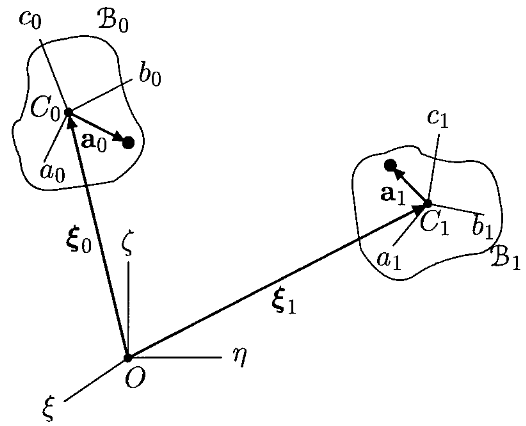


Figure B.1 Two-Body Configuration

where $\boldsymbol{\xi}_{ij}$ is the inertial position of C_i, the origin of frame $\mathcal{F}_{b_i}$, relative to C_j, the origin of frame $\mathcal{F}_{b_j}$ ($\boldsymbol{\xi}_{ij} = \boldsymbol{\xi}_i - \boldsymbol{\xi}_j$) and $\mathbf{B}_i$ is the transformation matrix from body frame $\mathcal{F}_{b_i}$ to the inertial frame $\mathcal{F}_i$. The matrix $\mathbf{B}_i$ can be represented as a function of the Euler angles $\boldsymbol{\theta}_i = (\psi_i, \theta_i, \phi_i)$. The configuration for two bodies is shown in Figure B.1. We limit consideration to those values of $\boldsymbol{\xi}_i$ and $\boldsymbol{\theta}_i$ for which the bodies are not in contact. The total potential function for a closed system of n rigid bodies is then

$$V(\boldsymbol{\xi}, \boldsymbol{\theta}) = \sum_{i<j} V_{ij} \qquad \boldsymbol{\xi} = (\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{n-1}),\ \boldsymbol{\theta} = (\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_{n-1}). \tag{B.6}$$

Note that this does not include gravitational interactions of particles within the same body. We assume these are exactly counteracted by internal forces to maintain the rigid body assumption.

B.2.1 A Special Case. We consider a particular closed system consisting of two rigid bodies $\mathcal{B}_0$ and $\mathcal{B}_1$. Furthermore, we require $\mathcal{B}_0$ to have a spherically symmetric mass distribution and we choose the body frame $\mathcal{F}_{b_0}$ so that the origin C_0 is the center of mass

of $\mathcal{B}_0$. From the previous section, the potential is

$$V(\boldsymbol{\xi}, \boldsymbol{\theta}) = -G \int_{\mathcal{B}_0} dm_0 \int_{\mathcal{B}_1} dm_1 \frac{1}{|\boldsymbol{\xi}_{10} + \mathbf{B}_1 \mathbf{a}_1 - \mathbf{B}_0 j \mathbf{a}_0|} \qquad \boldsymbol{\xi} = (\boldsymbol{\xi}_0, \boldsymbol{\xi}_1),\ \boldsymbol{\theta} = (\boldsymbol{\theta}_0, \boldsymbol{\theta}_1). \tag{B.7}$$

Because of the spherical symmetry requirement, we may accomplish the integration over body $\mathcal{B}_0$ (by introducing spherical coordinates in the body frame) to find

$$V(\boldsymbol{\xi}, \boldsymbol{\theta}_1) = -Gm_0 \int_{\mathcal{B}_1} \frac{1}{|\boldsymbol{\xi}_{10} + \mathbf{B}_1 \mathbf{a}_1|} dm_1. \tag{B.8}$$

That is, the result is the same as if all the mass of $\mathcal{B}_0$ were concentrated at its center of mass, as was first shown by Newton [78], and is independent of the attitude of $\mathcal{B}_0$. We may redefine the potential as a function of the relative position. Let $\boldsymbol{\lambda}_1 = \boldsymbol{\xi}_{10} = \boldsymbol{\xi}_1 - \boldsymbol{\xi}_0$. Then

$$V(\boldsymbol{\lambda}_1, \boldsymbol{\theta}_1) = -Gm_0 \int_{\mathcal{B}_1} \frac{1}{|\boldsymbol{\lambda}_1 + \mathbf{B}_1 \mathbf{a}_1|} dm_1. \tag{B.9}$$

Furthermore, we may now express all quantities in the body frame so that

$$V(\boldsymbol{\Lambda}_1) = -Gm_0 \int_{\mathcal{B}_1} \frac{1}{|\boldsymbol{\Lambda}_1 + \mathbf{a}_1|} dm_1 \tag{B.10}$$

and the Euler angle dependence is eliminated from the potential.

B.3 A Rigid Body in a Central Gravitational Field

Consider the special case of the previous section under the additional restriction that the mass of the spherically symmetric body $\mathcal{B}_0$ is much larger than that of the other. We refer to $\mathcal{B}_0$ as the *primary* and $\mathcal{B}_1$ as the *satellite.* Hence, we assume the center of mass of the two-body system is located at the center of mass of the primary and study the motion of the satellite relative to the primary. This is the *restricted two-body problem* and is equivalent to the motion of a rigid body in a central gravitational field with the center of

attraction located at the center of mass of the primary. We fix the origin O of the inertial frame at the center of attraction. The potential for a rigid body in this field is given by

$$V(\mathbf{\Lambda}) = -\int_{\mathcal{B}} \frac{G_*}{|\mathbf{\Lambda} + \mathbf{a}|} dm \tag{B.11}$$

where $G_* = Gm_0$ is the gravitational constant for the primary body. Here $\mathbf{\Lambda}$ is the vector from the center of attraction to the origin C of the body frame $\mathfrak{F}_b$ expressed in the body frame. The "1" subscript is omitted since we no longer need to distinguish bodies.

B.3.1 Series Expansion of the Potential. We may expand the integrand in Equation (B.11) using

$$\begin{aligned} |\mathbf{\Lambda} + \mathbf{a}|^{-1} &= [(\mathbf{\Lambda} + \mathbf{a}) \cdot (\mathbf{\Lambda} + \mathbf{a})]^{-1/2} \\ &= \left[|\mathbf{\Lambda}|^2 + 2\mathbf{a} \cdot \mathbf{\Lambda} + |\mathbf{a}|^2\right]^{-1/2} \\ &= |\mathbf{\Lambda}|^{-1} \left[1 + \frac{2\mathbf{a} \cdot \mathbf{\Lambda}}{|\mathbf{\Lambda}|^2} + \left(\frac{|\mathbf{a}|}{|\mathbf{\Lambda}|}\right)^2\right]^{-1/2} . \end{aligned} \tag{B.12}$$

Let $\varepsilon = |\mathbf{a}| \,/\, |\mathbf{\Lambda}|$ and $\cos\phi = (\mathbf{a} \cdot \mathbf{\Lambda})/(|\mathbf{a}|\,|\mathbf{\Lambda}|)$. Then

$$|\mathbf{\Lambda} + \mathbf{a}|^{-1} = |\mathbf{\Lambda}|^{-1} \left[1 + 2\varepsilon\cos\phi + \varepsilon^2\right]^{-1/2} . \tag{B.13}$$

For $\left|2\varepsilon\cos\phi + \varepsilon^2\right| < 1$ we can expand the term in square brackets in a binomial series and collect on powers of ε to get

$$|\mathbf{\Lambda} + \mathbf{a}|^{-1} = |\mathbf{\Lambda}|^{-1} \left[1 - \varepsilon\cos\phi - \frac{\varepsilon^2}{2}\left(1 - 3\cos^2\phi\right) + \mathcal{O}\left(\varepsilon^3\right)\right] . \tag{B.14}$$

This expansion may also be expressed as

$$|\mathbf{\Lambda} + \mathbf{a}|^{-1} = |\mathbf{\Lambda}|^{-1} \sum_{k=0}^{\infty} (-\varepsilon)^k P_k(\cos\phi) \tag{B.15}$$

where the P_k are the Legendre polynomials. Then the potential is written as

$$V(\mathbf{\Lambda}) = -\int_{\mathcal{B}} \frac{G_*}{|\mathbf{\Lambda}|} \left[\sum_{k=0}^{\infty} (-\varepsilon)^k P_k(\cos\phi) \right] dm. \tag{B.16}$$

Let $\mathcal{S}$ be a sphere of radius r centered on C such that the body $\mathcal{B}$ lies entirely within $\mathcal{S}$. Then for all $\mathbf{\Lambda}$ such that $|\mathbf{\Lambda}| > r$, the series expansion is absolutely and uniformly convergent [60:83] and may be integrated term by term to give

$$V(\mathbf{\Lambda}) = -\frac{G_*}{|\mathbf{\Lambda}|} \sum_{k=0}^{\infty} \left[\int_{\mathcal{B}} (-\varepsilon)^k P_k(\cos\phi) dm \right]. \tag{B.17}$$

This expression for the potential should be compared to the more typical development in astrodynamics in which a particle moves about a rigid body of arbitrary shape.[2]

For $\varepsilon \ll 1$ the higher-order terms in Equation (B.17) quickly approach zero and it seems reasonable to truncate the series. We refer to a particular truncation of the potential as the *nth-order potential approximation* when it includes terms up to $\mathcal{O}(\varepsilon^n)$. We denote the nth-order approximation of the potential as V_n. Thus,

$$V_n(\mathbf{\Lambda}) = -\frac{G_*}{|\mathbf{\Lambda}|} \sum_{k=0}^{n} \left[\int_{\mathcal{B}} (-\varepsilon)^k P_k(\cos\phi) dm \right]. \tag{B.18}$$

Likewise, the associated approximate model of rigid body motion will be called the *nth-order model.*

B.3.2 Force and Torque. The force and torque acting on the rigid body are derived from the potential. We begin by considering the force acting on a differential mass element dm. The potential for this element, from Equation (B.11), is given by

$$dV = -\frac{G_*}{R} dm \tag{B.19}$$

[2] See, *e.g.*, Battin [10:401–2].

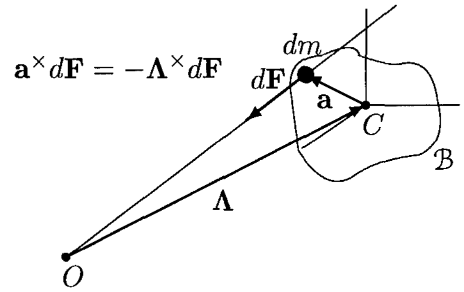


Figure B.2 Contribution to Torque About the Center of Mass

where $\mathbf{R} = \mathbf{\Lambda} + \mathbf{a}$ and $R = |\mathbf{R}|$. Let $\mathbf{F} = \mathbf{B}^{\mathsf{T}}\mathbf{f}$ and $\mathbf{G}_c = \mathbf{B}^{\mathsf{T}}\mathbf{g}_c$ be the total force and the torque about the point C expressed in the body frame. It should be clear that the force on the differential mass element can be expressed as

$$d\mathbf{F}(\mathbf{R}) = -\nabla_R dV(\mathbf{R}) = -\frac{G_*}{R^3}\mathbf{R}dm. \tag{B.20}$$

With a simple change of variables, it follows that $d\mathbf{F}(\mathbf{\Lambda}) = -\nabla_\Lambda dV(\mathbf{\Lambda})$. The contribution to the torque about C is $d\mathbf{G}_c(\mathbf{\Lambda}) = \mathbf{a}^\times d\mathbf{F}(\mathbf{\Lambda}) = -\mathbf{\Lambda}^\times d\mathbf{F}(\mathbf{\Lambda})$. The last equality follows from the alignment of $d\mathbf{F}$ with $\mathbf{R}$ as shown in Figure B.2. Integration of the above expressions over the rigid body gives the total force and moment about C. Exchanging the order of differentiation and integration, we find

$$\mathbf{F}(\mathbf{\Lambda}) = -\nabla_\Lambda V(\mathbf{\Lambda}) \tag{B.21}$$

$$\mathbf{G}_c(\mathbf{\Lambda}) = \mathbf{\Lambda}^\times \nabla_\Lambda V(\mathbf{\Lambda}). \tag{B.22}$$

Note that the torque is always orthogonal to $\mathbf{\Lambda}$ so that there is no torque component in the radial direction.

For some problems we consider, the distance $|\mathbf{\Lambda}|$ is fixed. In those instances we are only concerned with the torque and it is more convenient to work with the unit vector in the radial direction, $\boldsymbol{\gamma}$, rather than $\mathbf{\Lambda}$, as shown in Figure B.3. Again, a change of variables in

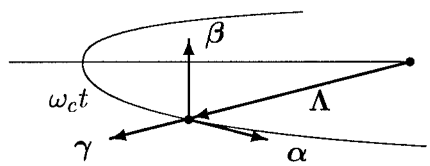


Figure B.3 The Orbital Frame in Body Frame Variables

Equation (B.22) gives

$$\mathbf{G}_c(\gamma) = \gamma^\times \nabla_\gamma V(\gamma). \tag{B.23}$$

This expression for the gravity torque was given in scalar form by Beletskii [15:10]. The expressions for force and torque given in Equations (B.21), (B.22), and (B.23) are relatively simple expressions in terms of the potential and allow a direct substitution of any order approximation.

B.3.3 Some Simple Approximations. For a given rigid body, the integral in Equation (B.16) is generally not solvable in closed form. However, the individual integrals in Equation (B.17) are solvable in closed form where the nth integral may be expressed in terms of the inertia integrals of order n.[3] Then the approximations introduced above allow us to circumvent the problem by expressing the potential as a finite sum of inertia integrals. In practice, very few terms are actually required in order to capture the dominant effects. Here we will consider approximations of the potential up to second order. We will need the Legendre polynomials

$$P_0(\nu) = 1 \tag{B.24a}$$

$$P_1(\nu) = \nu \tag{B.24b}$$

$$P_2(\nu) = \frac{1}{2}(3\nu^2 - 1) \tag{B.24c}$$

[3]The inertia integrals of order n are all integrals of the form $J_{a^i b^j c^k} = \int a^i b^j c^k dm$ where $i + j + k = n$. The first-order inertia integrals are just the components of the position of the center of mass while the second-order inertia integrals are related to the moments of inertia.

in order to evaluate the integrals.

Beginning with the zeroth-order approximation, we have

$$V_0(\mathbf{\Lambda}) = -\frac{G_*}{|\mathbf{\Lambda}|}\int_{\mathcal{B}} dm = -\frac{G_* m}{|\mathbf{\Lambda}|}. \tag{B.25}$$

This is precisely the potential for a particle of equal mass located at the point C. The force for this approximation is

$$F_0(\mathbf{\Lambda}) = -\nabla_\Lambda V_0(\mathbf{\Lambda}) = -\frac{G_* m}{|\mathbf{\Lambda}|^3}\mathbf{\Lambda} \tag{B.26}$$

and there is no resulting torque about C since

$$G_0(\mathbf{\Lambda}) = \mathbf{\Lambda}^\times \nabla_\Lambda V_0(\mathbf{\Lambda}) = -\frac{G_* m}{|\mathbf{\Lambda}|^3}\mathbf{\Lambda}^\times\mathbf{\Lambda} = 0. \tag{B.27}$$

Clearly, this approximation is of little use if we wish to study the attitude dynamics.

The next order of approximation is

$$\begin{aligned} V_1(\mathbf{\Lambda}) &= V_0(\mathbf{\Lambda}) + \frac{G_*}{|\mathbf{\Lambda}|}\int_{\mathcal{B}} \varepsilon\cos\phi\, dm \\ &= -\frac{G_* m}{|\mathbf{\Lambda}|} + \frac{G_*}{|\mathbf{\Lambda}|^3}\mathbf{\Lambda}\cdot\int_{\mathcal{B}} \mathbf{a}\, dm \\ &= -\frac{G_* m}{|\mathbf{\Lambda}|^3}\mathbf{\Lambda}\cdot\mathbf{\Lambda} + \frac{G_*}{|\mathbf{\Lambda}|^3}\mathbf{\Lambda}\cdot\boldsymbol{\chi} \\ &= -\frac{G_* m}{|\mathbf{\Lambda}|^3}\mathbf{\Lambda}\cdot(\mathbf{\Lambda}-\boldsymbol{\chi}) \end{aligned} \tag{B.28}$$

where $\boldsymbol{\chi}$ is the position of the center of mass in frame $\mathcal{F}_b$. Note that if $\mathcal{F}_b$ is a centroidal frame (*i.e.*, the origin C is the center of mass) then $V_1(\mathbf{\Lambda}) = V_0(\mathbf{\Lambda})$. The force for this approximation is

$$\begin{aligned} F_1(\mathbf{\Lambda}) &= -\nabla_\Lambda V_1(\mathbf{\Lambda}) \\ &= \frac{G_* m}{|\mathbf{\Lambda}|^3}(2\mathbf{\Lambda}-\boldsymbol{\chi}) - 3\frac{G_* m}{|\mathbf{\Lambda}|^5}\mathbf{\Lambda}\cdot(\mathbf{\Lambda}-\boldsymbol{\chi})\mathbf{\Lambda} \\ &= -\frac{G_* m}{|\mathbf{\Lambda}|^3}(\mathbf{\Lambda}+\boldsymbol{\chi}) + 3\frac{G_* m}{|\mathbf{\Lambda}|^5}(\mathbf{\Lambda}\cdot\boldsymbol{\chi})\mathbf{\Lambda}. \end{aligned} \tag{B.29}$$

This is *not* the same force as would be experienced by a particle of equal mass located at the center of mass (unless $\chi = \mathbf{0}$). The torque about C is

$$G_{c_1}(\mathbf{\Lambda}) = \mathbf{\Lambda}^{\times}\nabla_{\Lambda}V_1(\mathbf{\Lambda}) = \frac{G_* m}{|\mathbf{\Lambda}|^3}\mathbf{\Lambda}^{\times}\chi. \tag{B.30}$$

For a problem in which $|\mathbf{\Lambda}|$ is fixed, we have $\gamma = \mathbf{\Lambda}/|\mathbf{\Lambda}|$ and

$$G_{c_1}(\gamma) = \frac{G_* m}{|\mathbf{\Lambda}|^2}\gamma^{\times}\chi. \tag{B.31}$$

We refer to this as the *heavy-top torque* since it is the torque which acts on a rigid body moving about a point of the body that is fixed in a uniform gravitational field of strength $g = G_*/|\mathbf{\Lambda}|^2$ — the classical heavy top problem. Again, in this approximation no torque is present about C when the body frame is centroidal.

Finally, we consider the second-order approximation. This is the most important and useful approximation because it introduces the principal torque effects due to the central field while requiring no higher than second-order inertia integrals. The torque is referred to as the gravity-gradient torque since it arises from the linear variation in gravity across the body. The potential approximation is

$$V_2(\mathbf{\Lambda}) = V_1(\mathbf{\Lambda}) - \frac{G_*}{|\mathbf{\Lambda}|}\int_{\mathcal{B}} \frac{1}{2}\varepsilon^2(3\cos^2\phi - 1)dm \tag{B.32}$$

which reduces to

$$V_2(\mathbf{\Lambda}) = -\frac{G_* m}{|\mathbf{\Lambda}|} + \frac{G_*\mathbf{\Lambda}\cdot\chi}{|\mathbf{\Lambda}|^3} - \frac{1}{2}\frac{G_*\,\mathrm{tr}\,(\mathbf{I})}{|\mathbf{\Lambda}|^3} + \frac{3}{2}\frac{G_*\mathbf{\Lambda}\cdot\mathbf{I}\mathbf{\Lambda}}{|\mathbf{\Lambda}|^5}. \tag{B.33}$$

The above reduction requires use of the definition of the inertia matrix

$$\mathbf{I} = \int_{\mathcal{B}} (|\mathbf{a}|^2\,\mathbf{1} - \mathbf{a}\mathbf{a}^{\mathsf{T}})dm \tag{B.34}$$

along with the property

$$\operatorname{tr}(\mathbf{I}) = 2\int_{\mathcal{B}} |\mathbf{a}|^2 \, dm. \tag{B.35}$$

In terms of $\boldsymbol{\gamma}$ for fixed $|\boldsymbol{\Lambda}|$, the potential is

$$V_2(\boldsymbol{\gamma}) = -\frac{G_* m}{|\boldsymbol{\Lambda}|} + \frac{G_* \boldsymbol{\gamma}\cdot\boldsymbol{\chi}}{|\boldsymbol{\Lambda}|^2} - \frac{1}{2}\frac{G_* \operatorname{tr}(\mathbf{I})}{|\boldsymbol{\Lambda}|^3} + \frac{3}{2}\frac{G_* \boldsymbol{\gamma}\cdot\mathbf{I}\boldsymbol{\gamma}}{|\boldsymbol{\Lambda}|^3}. \tag{B.36}$$

Treating the case of variable $|\boldsymbol{\Lambda}|$ first, we have

$$\begin{aligned} F_2(\boldsymbol{\Lambda}) &= -\nabla_{\Lambda} V_2(\boldsymbol{\Lambda}) \\ &= -\frac{G_* m}{|\boldsymbol{\Lambda}|^3}(\boldsymbol{\Lambda}+\boldsymbol{\chi}) + 3\frac{G_* m}{|\boldsymbol{\Lambda}|^5}(\boldsymbol{\Lambda}\cdot\boldsymbol{\chi})\boldsymbol{\Lambda} \\ &\quad - \frac{3}{2}\frac{G_* \operatorname{tr}(\mathbf{I})}{|\boldsymbol{\Lambda}|^5}\boldsymbol{\Lambda} - 3\frac{G_*}{|\boldsymbol{\Lambda}|^5}\mathbf{I}\boldsymbol{\Lambda} + \frac{15}{2}\frac{G_* \boldsymbol{\Lambda}\cdot\mathbf{I}\boldsymbol{\Lambda}}{|\boldsymbol{\Lambda}|^7}\boldsymbol{\Lambda}. \end{aligned} \tag{B.37}$$

The torque for this approximation is

$$G_{c_2}(\boldsymbol{\Lambda}) = \boldsymbol{\Lambda}^{\times}\nabla_{\Lambda} V_2(\boldsymbol{\Lambda}) = \frac{G_* m}{|\boldsymbol{\Lambda}|^3}\boldsymbol{\Lambda}^{\times}\boldsymbol{\chi} + 3\frac{G_*}{|\boldsymbol{\Lambda}|^5}\boldsymbol{\Lambda}^{\times}\mathbf{I}\boldsymbol{\Lambda}. \tag{B.38}$$

Similarly, in the fixed radius case, we have

$$G_{c_2}(\boldsymbol{\gamma}) = \boldsymbol{\gamma}^{\times}\nabla_{\Lambda} V_2(\boldsymbol{\gamma}) = \frac{G_* m}{|\boldsymbol{\Lambda}|^2}\boldsymbol{\gamma}^{\times}\boldsymbol{\chi} + 3\frac{G_*}{|\boldsymbol{\Lambda}|^3}\boldsymbol{\gamma}^{\times}\mathbf{I}\boldsymbol{\gamma}. \tag{B.39}$$

Even if the body frame is centroidal, the second term in Equations (B.38) and (B.39) remains. This is the gravity-gradient torque. For a tri-inertial body in a centroidal frame, this term dominates the actual torque from the full expansion. However, for bodies with two or more nearly equal principal inertias, higher-order terms may be required. For a discussion of the need for higher-order inertia integrals, see Meirovitch [69].

Appendix C. Proof of Identity $\boldsymbol{\beta}^{\times}\mathbf{I} + \mathbf{I}\boldsymbol{\beta}^{\times} \equiv \{[\mathrm{tr}\,(\mathbf{I})\,\mathbf{1} - \mathbf{I}]\,\boldsymbol{\beta}\}^{\times}$

This appendix proves the stated identity for vector $\boldsymbol{\beta}$ and matrix $\mathbf{I}$. The only restriction is that $\mathbf{I}$ must be symmetric. We use indicial notation[1] for brevity.

It is well-known that any square matrix $\mathbf{A}$ may be split into its symmetric and skew-symmetric parts by the formulas

$$\mathbf{A}_{\mathsf{sym}} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathsf{T}}) \tag{C.1a}$$

$$\mathbf{A}_{\mathsf{skew}} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathsf{T}}). \tag{C.1b}$$

It then follows that a square matrix minus its transpose is always skew symmetric. Since

$$(\boldsymbol{\beta}^{\times}\mathbf{I})^{\mathsf{T}} = -\mathbf{I}\boldsymbol{\beta}^{\times} \tag{C.2}$$

we are assured that the sum on the left side of our identity is a skew-symmetric form. It only remains to be shown that it is the skew-symmetric form given on the right-hand side.

Before we address the primary problem, we first prove a general proposition on the form of a matrix minus its transpose. We make use of the Kronecker delta,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{C.3}$$

the permutation symbol,

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0 & \text{otherwise,} \end{cases} \tag{C.4}$$

[1] See, *e.g.*, Chapter 1 of Byron and Fuller [18].

and the identity

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}. \tag{C.5}$$

Proposition C.1. *Given a matrix* $\mathbf{A} = A_{ik}$*, the difference of the matrix and its transpose may be written as the skew-symmetric form* $\varepsilon_{ijk}b_j$ *where the vector* $\mathbf{b} = b_j$ *is given by* $b_j = \varepsilon_{ljm}A_{lm}$*. Alternatively, we say* $\mathbf{b}^\times = \mathbf{A} - \mathbf{A}^\mathsf{T}$*.*

Proof.

$$\begin{aligned} A_{ik} - A_{ki} &= \delta_{li}\delta_{mk}A_{lm} - \delta_{lk}\delta_{mi}A_{lm} \\ &= (\delta_{li}\delta_{mk} - \delta_{lk}\delta_{mi})A_{lm} \\ &= \varepsilon_{jik}\varepsilon_{jlm}A_{lm} \\ &= \varepsilon_{ijk}\varepsilon_{ljm}A_{lm} \\ &= \varepsilon_{ijk}b_j. \end{aligned}$$

□

We now proceed to our main result.

Proposition C.2. *A vector* $\boldsymbol{\beta}$ *and a symmetric matrix* $\mathbf{I}$ *satisfy the identity*

$$\boldsymbol{\beta}^\times\mathbf{I} + \mathbf{I}\boldsymbol{\beta}^\times \equiv \{[\mathrm{tr}\,(\mathbf{I})\,\mathbf{1} - \mathbf{I}]\,\boldsymbol{\beta}\}^\times. \tag{C.6}$$

Proof. Let $\boldsymbol{\beta} = \beta_j$ and $\mathbf{I} = I_{ij}$. We then have

$$\begin{aligned} \boldsymbol{\beta}^\times\mathbf{I} + \mathbf{I}\boldsymbol{\beta}^\times &= \varepsilon_{ijk}\beta_j I_{kl} + I_{ik}\varepsilon_{kjl}\beta_j \\ &= \varepsilon_{ijk}\beta_j I_{kl} - \varepsilon_{ljk}\beta_j I_{ki}. \end{aligned}$$

Let $A_{il} = \varepsilon_{ijk}\beta_j I_{kl}$. Then

$$
\begin{aligned}
\boldsymbol{\beta}^{\times}\mathbf{I} + \mathbf{I}\boldsymbol{\beta}^{\times} &= A_{il} - A_{li} \\
&= \varepsilon_{iml}\varepsilon_{nmo}A_{no} \quad \text{(by the previous proposition)} \\
&= \varepsilon_{iml}\varepsilon_{nmo}\varepsilon_{njk}\beta_j I_{ko} \\
&= \varepsilon_{iml}(\delta_{mj}\delta_{ko} - \delta_{mk}\delta_{jo})\beta_j I_{ko} \\
&= \varepsilon_{iml}(I_{kk}\delta_{mj} - I_{mj})\beta_j \\
&= \{[\mathrm{tr}\,(\mathbf{I})\,\mathbf{1} - \mathbf{I}]\,\boldsymbol{\beta}\}^{\times}.
\end{aligned}
$$

□

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Vita

Major Jeffrey A. Beck [redacted], and is a 1979 graduate of North Penn High School in Lansdale, Pennsylvania. He earned his bachelors degree in mechanical engineering from Lehigh University in 1984 at which time he received a commission in the Air Force through ROTC. His initial assignment was with the Air Force Flight Test Center at Edwards Air Force Base where he designed and installed test instrumentation for numerous test programs including the Air-Launched Cruise Missile, F-15, F-16, MC-130, and HH-60 programs. From 1988-89 he completed his masters program in aeronautical engineering at the Air Force Institute of Technology, specializing in aerodynamics and flight control. He was then assigned to the Flight Dynamics Directorate of Wright Laboratory where he conducted an in-house research program in aircraft agility and manueverability. In 1992, he returned to the Air Force Institute of Technology for his PhD program under the direction of Dr Chris Hall. He was assigned to Phillips Laboratory in 1995 where he currently serves as chief of the Space Systems Guidance, Navigation, and Control Branch.

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We apply noncanonical Hamiltonian methods to examine relative equilibria of a rigid body in a central gravitational field. These equilibria correspond to fixed points of a reduced set of equations expressed in a rotating frame and are representative of an orbiting satellite with fixed attitude relative to an observer rotating at the orbital rate. Our objective is to clarify the relationship between the classical approximation and a recent noncanonical Hamiltonian treatment. In contrast to the classical approximation, the orbital and attitude equations of motion for the noncanonical system remain coupled and the general solution is a circular orbit for which the orbit center and the center of attraction are not necessarily coincident. Our approach involves development of a hierarchy of Hamiltonian approximations. The hierarchy consists of the existing noncanonical system and two noncanonical formulations which we derive for rigid bodies subject to certain constraints --- motion about a fixed point and motion about a point following a Keplerian orbit. The classical solution is dynamically equivalent to this latter constrained (Keplerian) system. We apply Hamiltonian methods to identify relative equilibria and determine stability conditions. In general, we find that relative equilibria for the Keplerian and unconstrained systems are in close agreement.

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